Financial Interactions and Collective States: Part III. Description of Dynamic Collective States

Pierre Gosselin* Aïleen Lotz†

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Abstract

In a series of papers, we analyzed the mechanisms of capital allocation and accumulation among firms, investors, and banks within the framework of Field Economics. At the macro level, we found that multiple collective states may emerge, each organized in groups of agents characterized by distinct levels of disposable capital, returns, and stakes across sectors. This multiplicity of collective states reflects the inherent instability and default-prone nature of financial markets. It also underscores the necessity of transitions among these states.

These transitions were not fully accounted for in our previous description of collective states, where we only considered marginal fluctuations around equilibrium levels. To fill this gap, the present paper aims to characterize the possible dynamic mechanisms underlying transitions between these collective states. It reviews and extends our previous analysis by examining their probability and stability. By introducing the notion of dynamic sub-collective states, it explores potential patterns of transition—such as morphing, merging, or splitting—among collective states. This study thus provides the foundational tools and analytical framework that will be employed to develop a field-theoretic description of dynamic collective states.

Key words: Financial Markets, Real Economy, Capital Allocation, Statistical Field Theory, Background fields, Collective states, Multi-Agent Model, Interactions.

JEL Classification: B40, C02, C60, E00, E1, G10

1 Introduction

In previous work, we studied the mechanisms of capital allocation among a large number of investors, firms, and banks in a sector space through the notion of collective states. As a first approximation, a collective state is decomposed into independent groups of sectors, each characterized by specific levels of cross-sectoral stakes, disposable capital, and sectoral returns. These variables jointly define the phase the is group in. We refer to a group in a given phase as a sub-collective state. A collective state is therefore defined as a collection of sub-collective states.

Since there exists an infinite number of possible decompositions of the sector space into groups, and since each group may occupy several distinct phases, the set of sub-collective states is infinite. As a consequence, the number of collective states is itself infinite.

Both sub-collective states and the collective states they compose are inherently prone to instability. Even small perturbations in the level of disposable capital or returns within a given sector may propagate to other sectors, triggering adjustments in their capital allocation and returns, which

*Pierre Gosselin : Institut Fourier, UMR 5582 CNRS-UGA, Université Grenoble Alpes, BP 74, 38402 St Martin d'Hères, France. E-Mail: Pierre.Gosselin@univ-grenoble-alpes.fr

†Aïleen Lotz: Cerca Trova, BP 114, 38001 Grenoble Cedex 1, France. E-mail: a.lotz@cercatrova.eu

in turn amplify the initial perturbation. Consequently, neither sub-collective nor collective states can be regarded as independent or isolated. Instead, they belong to a fully interconnected system of interacting states, characterized by nonlinear propagation mechanisms and feedback effects.

Within this system, states undergo medium- to long-term transformations, including transitions, recomposition, and defaults, through which the system evolves from one collective state to another.

Accounting for the multiplicity of sub-collective states, their interactions, and the transitions between them requires a specific formalism. Before introducing such a framework, however, it is necessary to identify the properties it must satisfy. To this end, the present paper consolidates our previous results, provides a systematic characterization of sub-collective states and their associated default configurations, and extends the analysis of their stability. We then broaden this description to incorporate dynamical deviations around sub-collective states, which we refer to as *deviation states*. Within this broader perspective, a collective state can be understood as the combination of a *basis state*—describing a given partition of stakes among investors, banks, and firms—and *fiber states*, which encapsulate the multiple configurations of capital and returns compatible with that partition, as studied in Gosselin and Lotz (2025a,b). Two distinct types of fields must therefore be introduced: fields associated with basis states and fields associated with fiber states.

The formalism we aim to develop is a field-based model in which deviation states constitute the fundamental variables. Within this framework, a given group of interconnected investors may undergo multiple variations and transitions, switching from one phase to another, potentially leading to default. Sub-collective states may interact, merge, or reorganize, implying that the sectoral composition of a group is itself endogenous. As a consequence, sectors may appear or disappear within a group, leading to transitions between collective states in which the decomposition into groups is modified, as demonstrated in Gosselin and Lotz (2023d). Transitions between collective states may be induced either by fiber or basis transitions, and may occur through sequences of successive transformations involving both types of fields, each type of transition potentially triggering the other. This structure reveals that the system is inherently unstable: the permanent occurrence of discontinuous transitions generates an internal dynamics that continuously explores the infinite space of possible collective states, thereby endogenously realizing it.

Based on these properties, we propose a preliminary formalism, which will be fully developed in subsequent work. In this framework, each group is modeled by a field that encodes both the set of the possible phases of the group and the fluctuations around these phases. The resulting formalism therefore involves an infinite number of fields, corresponding to all possible decompositions of the sectoral space into groups of agents, together with their associated dynamics and transitions.

This construction leads to a model that is substantially broader than our original framework. It allows for the explicit treatment of interactions between relatively independent groups, their consequences in terms of transitions between sub-collective states, and, ultimately, transitions between collective states. In this sense, the proposed formalism synthesizes and extends the core features developed in our previous works[1] (Gosselin and Lotz (2024)).

This work is organized as follows. After a brief literature review in Section 2, Part I reviews and extends our previous results. In Section 3, we introduce the notion of collective states, and Section 4 describes these states in terms of their basis and fiber variables. Sections 5 and 6 study the probabilities and stability ranges of these states. Section 7 presents a compact notation for collective states that accounts for possible interactions between groups. Part II extends the framework to a fully dynamic model of collective states. Section 8 motivates the need for these extensions, Section 9 derives the possible forms of dynamic collective states, and Section 10 describes the expected patterns of transitions between them. In Section 11, we summarize the characteristics required of

[1] Gosselin and Lotz ().

the field-theoretic formalism to describe these transitions. Finally, Section 12 concludes the paper.

2 Litterature Review

Five major directions are related to our approach. The first direction addresses heterogeneity among agents through distributions of agents modeled by probability densities. In mean field games (MFG) and mean field control, individual agents are negligible compared with the population but interact through aggregate variables (see Bensoussan, Frehse, and Yam (2018) and Lasry and Lions (2007, 2010a, 2010b)). This approach has been applied in the dynamic programming framework developed by Gomes, Vilanova, and others (see Gomes et al., 2015; Achdou et al., 2014). Heterogeneous Agent New Keynesian (HANK) models incorporate similar heterogeneity into macroeconomic structures. An equilibrium probability distribution is derived from a set of optimizing heterogeneous agents in a new Keynesian context (see Kaplan and Violante (2018), as well as Kaplan, Moll, and Violante (2018) for quantitative implementations). Information-theoretic approaches build on Sims' (2003, 2006) rational inattention theory to model agents optimizing under informational constraints (see Yang (2018) and Matejka and McKay (2015)). This information theoretic approach considers probabilistic states around the equilibrium and replaces the Walrasian equilibrium with a statistical equilibrium derived from an entropy maximisation program. In these three types of models, probability distributions can be seen as particular types of collective states postulated *a priori* as equilibria of the microeconomics setup.

Field economics, on the contrary, builds on the interactions between agents at the microeconomic level. We do not postulate an equilibrium probability distribution for each type of agent. Rather we consider any possible probability density for the entire system of N agents and their interactions, and translate these probability densities in terms of fields. Since the fields encompass all the possible densities of agents as their realizations[2], the state-space in field economics is thus much larger than those considered in the above approaches. This allows us to study the agents' economic structural relations and the emergence of the collective states induced by these specific micro-relations, which in turn impact each agent's stochastic dynamics at the microeconomic level. These emerging collective states are in general multiple with their own characteristics, average quantities, and distributions of agents, etc.

Interacting agents with heterogeneous behavioral rules have been dealt with by multi-agent systems, particularly agent-based models (ABMs), with an emphasis on non-equilibrium dynamics and bounded rationality (Gaffard and Napoletano (2012), Delli Gatti et al. (2005)). Mandel, Landini, and Gallegati (2010, 2012) further develop ABMs to capture innovation, sectoral dynamics, and macroeconomic fluctuations. The field of economic networks, notably Jackson (2010, 2014), focuses on the structural properties of agent interactions within economic systems. Both approaches are highly numerical and model-dependent. They also rely on microeconomic relations, such as *ad hoc* reaction functions, that may be too simplistic. Field economics, on the contrary, accounts for transitions between scales. Macroeconomic patterns do not emerge solely from the dynamics of a large population of agents: they are grounded in behaviours and interaction structures. Describing these structures in terms of field theory allows for the emergence of phases at the macro scale, and the study of their impact at the individual level.

Econophysics applies statistical physics methods to socio-economic systems (see Abergel et al. (2011a, 2011b) for reviews of the field's developments). Lux (2008, 2016) explores stylized facts and agent-based dynamics, while Kleinert (2009) uses path integral formulations to analyze financial

[2] In our formalism, the notion of fields refers to some abstract complex functions defined on the state space and is similar to the second-quantized-wave functions of quantum theory.

markets. Other relevant works include Bardoscia et al. (2017), Bouchaud and Mézard (2000), Chakraborti et al. (2011), Chakraborti et al. (2013). However, Econophysics does not apply the full potential of field theory to economic systems. Instead, it seeks to reveal empirical laws in economic systems. Field economics, in contrast, keeps track of usual microeconomic concepts, such as utility functions, expectations, and forward-looking behaviors. It integrates these behaviours into the analytical treatment of multi-agent systems by translating the main characteristics of optimizing agents in terms of statistical systems. Closer to our approach, Bardoscia et al. (2017) study a general equilibrium model for a large economy in the context of statistical mechanics, and show that phase transitions may occur in the system. Our issue is similar, but our use of field theory addresses a larger class of dynamic models.

The second direction addresses interactions between Finance and the Real Economy. The interactions between financial frictions and capital accumulation have typically been modeled within DSGE frameworks, enriched with credit constraints, incomplete markets, or firm-level heterogeneity. Cochrane (2006) provides a review of asset pricing in macro models. Bernanke, Gertler, and Gilchrist (1999) model the financial accelerator, linking firm balance sheets to investment dynamics. Holmstrom and Tirole (1997) study liquidity provision under moral hazard. More recent work includes Campello et al. (2010) and Quadrini (2012). Other contributions extend this framework to heterogeneous agents and endogenous risk (Grassetti et al. (2022), Grosshans and Zeisberger (2018), Böhm, Kuehn, and Schmedders (2008), Khan and Thomas (2013), Monacelli et al. (2011) and Moll (2014)).

Field economics differs from DSGE models in that their models are micro models that stand for the entire set of agents. This does not allow the study of the diffusion and circulation of capital among agents. Field economics, on the contrary, studies the entire system of agents. When dealing with capital, it describes the different states that result from this diffusion and circulation of capital. Besides, the relative (un)stability of capital allocation can be assessed globally: it is the relative distribution of returns or external conditions that determine investors' allocation of capital to firms. We can study how a local interest rate or a change in sectoral returns would impact the equilibrium of one sector, and the whole system.

The third direction covers the literature on default and systemic risk, and their contagion within financial networks. Early theoretical models were developed by Allen and Gale (2000), Cifuentes, Ferrucci, and Shin (2005), and Gai and Kapadia (2010). The network-based approach has been formalized by Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) and extended by Bardoscia et al. (2019) and Glasserman and Young (2015) using graph-theoretic stress-testing techniques and feedback loop dynamics (see also Battiston et al. (2012, 2020) for recursive losses in interconnected systems and Haldane and May (2011) for an ecology and epidemiology approach to financial instability). Empirically, contributions include Reinhart and Rogoff (2009), Gennaioli, Martin, and Rossi (2012, 2018), Adrian and Brunnermeier (2016), Langfield, Liu, and Ota (2020).

This paper adds to these studies by considering firms, investors and banks in the analysis. Moreover, collective states are described both in terms of global averages and sectoral quantities, allowing disparities in firms' returns and borrowing conditions across agents to be explicitly accounted for. Our analysis therefore moves back and forth between the micro and macro levels to identify the conditions under which micro-level defaults propagate to the macroeconomic level.

A fourth strand of the literature emphasizes the intermediary role of banks and the regulatory frameworks—especially macroprudential policies—designed to stabilize the financial system. It includes theories of banking (Diamond and Dybvig (1983); Diamond and Rajan (2001); Freixas, Parigi and Rochet (2000); Freixas and Rochet (2008)) and modern macro-financial models (Gertler and Kiyotaki (2010); Brunnermeier and Sannikov (2014); Adrian and Brunnermeier (2016)). Macroprudential tools are explicitly aimed at reducing the likelihood that idiosyncratic shocks trigger widespread defaults. Empirical studies and policy evaluations (Borio and Drehmann (2009), Brunnermeier, Crockett, Goodhart, Persaud, and Shin (2009); Acharya and Richardson (2012); Galati and Moessner (2013); Drehmann and Juselius (2014); IMF (2014); Cerutti, Claessens and Laeven (2017)) indicate that well-designed macroprudential measures can lower the probability of default, although their effectiveness depends on timing, calibration, and cross-border coordination.

Our work differs from these approaches in that, in our framework, instability is inherent to collective states, and the apparently more capitalized, high-yield investors or banks are themselves sources of systemic fragility. Since risk is an ad-hoc concept rather than an absolute measure, restricting access to credit to low-return investors and firms in the name of prudence may in fact strengthen high-return investors and banks, thereby increasing structural instability.

The present work is specifically related to a fifth direction of research: the field formalism for collective states developped in Gosselin and Lotz (2023a,b,c,d). This litterature is specifically based on the assumption that collective states can be decomposed in sub-collective states, that is the combination of a group and the phase it is in. Using this assumption, an effective field formalism is developped that associates one specific field to each group of the decomposition along with all the possible phases this group can take. The number of possible groups composing the collective states of the system, being infinite, the number of fields in this formalism will also be infinite. In this context, the interactions between fields describe the transitions, modifications and creations of the groups The closest modeling tool to the one described here, although completely different in the objects studied would be string field theory, particularly its description in terms of an infinite family of fields (see Erbin 2018): string states can be seen as collections of large number of oscillators of multiple frequencies, and each field in this infinite family of fields correspond to one of these state and its potential variations and dynamics. The analogy with this branch of theoretical physics is only formal but stresses the fact that describing interactions between large number of emerging states needs an enlargment of the formalism, by introducing a sort of field of fields.

Collective States: Review and Extensions

In the context of field economics, multiple collective states may emerge. Each is composed of several groups of agents in a particular phase, characterized by their level of sectoral disposable capital and returns and cross-sectoral stakes. These collective states are not necessarily stable, and can experience transitions towards other collective states. In this part we gather our previous results and describe the characteristics of sub-collective states, including their possible default states. We also present and extend the study of the stability of sub-collective states. Ultimately, we introduce a compact notation for these states that accounts for their possible interactions.

3 Collective States

In view of our previous works on financial interactions, we present the main characteristics of emerging collective states in systems of financial interactions, detailed in the Appendices[3]. The system is composed of several fields, each describing the different types of agents and their main characteristics: disposable capital, sectors of location, and connections with other agents, here stakes. Besides, we can associate to each type of agent an action functional, which translates, in

[3]The model, the action functionals, a summary of the main equations and the principle of resolution is given in Appendix 1, while technical details are given in Appendices 2 to 4.

terms of fields, the agents' microeconomic behaviour. The system of these action functionals is, what we call, the field model. The field realizations that minimize these action functionals are called the background fields of the system and the set of background fields are called the collective states of the system.

3.1 Fields

There are two types of fields involved in the present model[4]: the three fields that describe capital distribution in each sector, among firms, investors, and banks, respectively; and the two fields representing the possible connections between agents of different sectors, i.e. the stakes, for investors and banks respectively.

3.1.1 Fields of capital

The fields describing the capital distribution among agents within each sector of the sector space are, respectively:

Investors

$$\hat{\Psi}\left(\hat{K}, X\right)$$

Banks

$$\bar{\Psi}\left(\bar{K}, X\right)$$

Firms

$$\Psi\left(K, X\right)$$

The arguments of these fields are the level of disposable capital per agent and sector, so that each realization of these fields $\hat{\Psi}$,$\bar{\Psi}$ and Ψ yields a distribution of disposable capital $\hat{K}$, $\bar{K}$ and K for investors, banks and firms respectively in each sector X.

The dynamical properties of the fields are summarized in their respective action functionals, $\hat{S}\left(\hat{\Psi}\right)$, $\bar{S}\left(\bar{\Psi}\right)$, and $S\left(\Psi\right)$ that describe the accumulation process of each type of agents, depending on their returns and their connections to other sectors[5].

3.1.2 Fields of stakes

The endogenous connections between agents are their respective stakes. In our setup firms are exlusively producers, and any investment activity of a firm is deemed as that of an investor. As a result, there will be only two distinct fields of stakes in our model, one for investors and one for banks[6].

Investors The field of stakes for investors is written as:

$$\Gamma\left(\hat{S}^{(T)}, X', X\right)$$

[4] See Gosselin and Lotz (2024, 2025a,b).

[5] The formula for the field action functionals are given in Appendix 1.

[6] See Gosselin and Lotz (2025a,b).

where X is the sector of origin of the stakes, and X' is the sector invested in, and $\hat{S}^{(T)}$ is the vector of all possible stakes of investors:

$$\hat{S}^{(T)} = \left(S_1, \hat{S}_1, S_2, \hat{S}_2\right)$$

In this vector, the shares and loans taken by an investor X in firms X are denoted S_1 and S_2 respectively, whereas the shares and loans taken by an investor X in an investor X' are denoted $\hat{S}_1$ and $\hat{S}_2$ respectively.

Banks: The field of stakes for banks is written as:

$$\bar{\Gamma}\left(\bar{S}^{(T)}, X', X\right)$$

where X is the sector of origin of the stakes, and X' is the sector invested in, and $\bar{S}^{(T)}$ is the vector of all possible stakes of banks:

$$\bar{S}^{(T)} = \left(\bar{S}_E, S_E^B, \hat{S}_E^B, \bar{S}_L, S_L^B, \hat{S}_L^B\right)$$

In this vector, the shares and loans taken by a bank X in firms X are denoted S_E^B and S_L^B respectively, whereas the shares and loans taken by a bank X in an investor X' are denoted $\hat{S}_E^B$ and $\hat{S}_L^B$ respectively and the shares and loans taken by a bank X in a bank X' are denoted $\bar{S}_E$ and $\bar{S}_L$ respectively.

The fields of stakes Γ and $\bar{\Gamma}$ describe the possible repartitions of investments' across sectors for investors and banks respectively and the dynamical aspects of these fields are described by the two action functionals $S(\Gamma)$ and $S(\bar{\Gamma})$[7].

3.2 Background Fields

The minimizations of the action functionals $S(\Psi)$, $\hat{S}\left(\hat{\Psi}\right)$ and $\bar{S}(\bar{\Psi})$ for disposable capital and $S(\Gamma)$ and $S(\bar{\Gamma})$ for stakes yield the background fields of the system, that are the most probable configurations for each field Ψ, $\hat{\Psi}$ and $\bar{\Psi}$, Γ and $\bar{\Gamma}$. These background fields are written $\hat{\Psi}_0(K,X)$, $\bar{\Psi}_0(K,X)$ and $\Psi_0(K,X)$, and describe the most likely distribution of capital for investors, banks and firms, whereas $\Gamma_{0,X',X}\left(\hat{S}^{(T)}\right)$ and $\Gamma_{0,X',X}\left(\bar{S}^{(T)}\right)$ represent the most likely distribution of stakes of investors and banks across sectors.

The complete collection of possible set of background fields:

$$\left\{\hat{\Psi}_0(K,X), \bar{\Psi}_0(K,X), \Psi_0(K,X), \Gamma_{0,X',X}\left(\hat{S}^{(T)}\right), \Gamma_{0,X',X}\left(\bar{S}^{(T)}\right)\right\}$$

define the collective states of the system. However, these fields are not independent. They are interconnected with each other. This reflects that disposable capital depends on stakes, and that stakes between sectors depend also on disposable capital and returns. We can show that these sets of background fields depend on average sectoral stakes[8] that solve the return equations of the system. These return equations were derived in Gosselin and Lotz (2025a,b)[9]. Therefore, these sectoral stakes become the fundamental variables of the system.

In each collective state, and for any quantity, such as stakes, returns and disposable capital, both overall and sectoral averages can be obtained.

[7] See Appendix 1, formula (95) and (97).
[8] See Appendix 1.
[9] See Appendix 2 for a summary.

3.2.1 Background fields of stakes

The minimization of the action functionals $S(\Gamma)$ and $S(\bar{\Gamma})$ yield the most probable realizations of the fields of stakes, the background fields. These background fields describe the distribution of stakes in each sector. They are gaussian distributions centered around the sectoral averages $\hat{S}_{\eta}^{(T)}(X', X)$ and $\bar{S}_{\eta}^{(T)}(X', X)$, that characterize the collective states. Since there exist several sectoral averages, there are several background fields and each of these background fields is defined by a set of values $\left\{\hat{S}_{\eta}^{(T)}(X', X), \bar{S}_{\eta}^{(T)}(X', X)\right\}$ across sectors.

Thus, solving the model and finding the collective states amounts to finding the possible sets $\left\{\hat{S}_{\eta}^{(T)}(X', X), \bar{S}_{\eta}^{(T)}(X', X)\right\}$.

Investors The background field of stakes for investors is given by:

$$\Gamma_{0,X',X}\left(\hat{S}^{(T)}\right) = N \exp\left(-\sum_{\eta} \frac{\left(\hat{S}_{\eta}^{(T)} - \hat{S}_{\eta}^{(T)}(X', X)\right)^2}{2\sigma_{\hat{K}}^2}\right) \tag{1}$$

where N is a normalization factor that describes the overall number of connections between investors.

Banks The background field of stakes for banks is given by:

$$\Gamma_{0,X',X}\left(\bar{S}^{(T)}\right) = \bar{N} \exp\left(-\sum_{\eta} \frac{\left(\bar{S}_{\eta}^{(T)} - \bar{S}_{\eta}^{(T)}(X', X)\right)^2}{2\sigma_{\hat{K}}^2}\right) \tag{2}$$

where $\bar{N}$ is a normalization factor that describes the overall number of connections between banks and between banks and investors.

3.2.2 Background fields of capital

The background fields $\bar{\Psi}$ and $\hat{\Psi}$ describe the distribution of capital in each sector, from which the average disposable capital per sector and per type of agent in the collective state can be derived. As for stakes, these values depend on the sectoral averages for stakes, so that the collective state can, in last analysis, be described in terms of stakes' sectoral averages.

The background fields $\Psi(K, X)$, $\hat{\Psi}(K, X)$, and $\bar{\Psi}(K, X)$ are expressed in terms of several types of variables. First, in terms of returns, for investors, denoted $\hat{f}(X)$, and for banks, denoted $\bar{f}(X)$. Second, in terms of the averages of these returns, denoted $\left\langle\hat{f}\right\rangle$ and $\left\langle\bar{f}\right\rangle$ respectively. Third, in terms of the sectoral stakes and their averages[10]. Finally, in terms of the average fields for investors and banks[11], $\left\|\hat{\Psi}\right\|^2$ and $\left\|\bar{\Psi}\right\|^2$. In first approximation, the formulas for the background fields can be written as follows:

Firms The background field for firms is given by:

$$|\Psi(K, X)|^2 = |\Psi_0(X)|^2 - \epsilon\left(\frac{\left(f_1^{(e)}(X) K - \bar{C}(X)\right)^2}{\sigma_{\hat{K}}^2} + \frac{f_1^{(e)}(X)}{2}\right) \tag{3}$$

[10] See Gosselin and Lotz (2025a, b).

[11] Computed in Gosselin and Lotz (2025a, b).

where $f_1^{(e)}(X)$ is the firm's net return of production once loans have been repaid, and $\bar{C}(X)$ is the average cost per unit of capital.

Investors The background field for investors is given by:

$$\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2 = \left\|\hat{\Psi}_0(X)\right\|^2 - \hat{\mu}\left\{\left(\frac{\hat{K}_1^2\hat{g}^2(X)}{2\sigma_{\hat{K}}^2}+\frac{\hat{g}(X)}{2}\right)-\left(\frac{\left\langle\hat{K}\right\rangle^2\langle\hat{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left\langle\hat{g}^{ef}\right\rangle\frac{\hat{K}}{\left\langle\hat{K}\right\rangle}\right\} \tag{4}$$

where $\hat{g}(X)$ is defined as:

$$\hat{g}(X) = \frac{\hat{f}(X)}{\langle\hat{f}\rangle}+\frac{\left\langle\hat{S}(X',X)\right\rangle_{X'}}{1-\left\langle\hat{S}\right\rangle}+\frac{\left(\left\langle\hat{S}_1^B(X,X')\right\rangle_{X'}+\left\langle\hat{S}_2^B(X,X')\right\rangle_{X'}+\frac{\langle\hat{S}_1^B\rangle+\langle\hat{S}_2^B\rangle}{1-\langle\hat{S}\rangle}\right)\frac{\langle\bar{K}\rangle\|\bar{\Psi}\|^2}{\langle\hat{K}\rangle\|\hat{\Psi}\|^2}\langle\bar{f}\rangle}{\left(1-\langle\bar{S}\rangle\right)\langle\bar{f}\rangle}$$

The coefficients $\langle\hat{g}^{ef}\rangle$ is an average coefficient that measures the impact on returns of any modification in the field[12].

Banks The background field for banks is given by:

$$\left|\bar{\Psi}\left(\bar{K},X\right)\right|^2 = \left|\bar{\Psi}_0(X)\right|^2 - \hat{\mu}\left\{\left(\frac{\bar{K}_1^2\bar{g}^2(X)}{\sigma_{\hat{K}}^2}+\frac{\bar{g}(X)}{2}\right)-\left(\frac{\left\langle\hat{K}\right\rangle^2\langle\hat{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left\langle\hat{g}^{Bef}\right\rangle\frac{\bar{K}}{\left\langle\hat{K}\right\rangle}\right\} \tag{5}$$

where:

$$\bar{g}(X) = \left(\left(1-\langle\bar{S}\rangle\right)\bar{f}(X)+\left\langle\bar{S}(X',X)\right\rangle_{X'}\langle\bar{f}\rangle\right)\frac{\left\langle\bar{S}(X',X)\right\rangle}{1-\left\langle\bar{S}(X',X)\right\rangle}$$

and $\langle\hat{g}^{Bef}\rangle$ measures the impact on returns of any modification in the field[13].

In the above expressions, the functions $\hat{g}(X)$ and $\bar{g}(X)$ are mixed returns composed of several factors: first, investors' and banks' returns $\hat{f}(X)$ and $\bar{f}(X)$ and their averages $\left\langle\hat{f}\right\rangle$ and $\langle\bar{f}\rangle$; second, the agents' stakes and their averages[14]; and third, the average fields[15] $\left\|\hat{\Psi}\right\|^2$ and $\|\bar{\Psi}\|^2$. Therefore, the functions $\hat{g}(X)$ and $\bar{g}(X)$ describe how investors' and banks' sectoral returns $\hat{f}(X)$ and $\bar{f}(X)$ spread and combine through the stakes $\hat{S}_\eta^{(T)}(X',X)$, $\bar{S}_\eta^{(T)}(X',X)$ and their respective averages.

Note that the coefficients $\langle\hat{g}^{ef}\rangle$ and $\langle\hat{g}^{Bef}\rangle$ are average coefficients that measure the impact on returns of any modification in the field[16].

3.3 Sub-Collective States

3.3.1 Collective States as Collections of Sub-Collective States

Under uncertainty, agents connect to only a finite number of neighbouring agents[17], so that agents are organized into several and loosely-connected groups. The background fields (1), (2), (3), (4),

[12] In general, these coefficients are negligible.
[13] In general, these coefficients are negligible.
[14] See Gosselin and Lotz (2025a, b).
[15] Computed in Gosselin and Lotz (2025a, b).
[16] In general, these coefficients are negligible.
[17] See Gosselin and Lotz (2025a,b),

(5), and the collective states they represent, must therefore be computed for any arbitrary group G of agents.

For any group [18] G, multiple solutions to the action functionals' minimization equations exist, each defining a background field and a corresponding phase. Therefore background fields are not unique.

Alternately, each phase of a group is defined by one set of background fields (1), (2), (3), (4), (5) among multiple possibilities. We will define a sub-collective state as a group within a specific phase.

The full collective states of the system will thus consist, first in a decomposition of the sector space in groups, and, second, in a collection of subcollective states associated to this decomposition.

3.3.2 Recovering collective state from subcollective states

Thus, each collective state is associated with an arbitrary decomposition of the sector space into a collection $\{G\}$ of several groups, each in a phase defined by its background fields:

$$\Gamma_{0,G}\left(\hat{S},X',X\right),\bar{\Gamma}_{0,G}\left(\bar{S},X',X\right),\hat{\Psi}_{0,G}\left(\hat{K},X\right),\bar{\Psi}_{0,G}\left(\bar{K},X\right),\Psi_{0,G}\left(K,X\right)$$

In this context, the background fields associated to a given decomposition of the whole sector space in groups $\{G\}$ are the product of the background fields for each group of the decomposition.

The collective state for the whole set of agents is thus defined by the product of fields of stakes stakes for investors and banks respectively:

$$\prod_G \Gamma_{0,G}\left(\hat{S},X',X\right),\ \prod_G \bar{\Gamma}_{0,G}\left(\bar{S},X',X\right) \tag{6}$$

and by the product of the fields of disposable capital for firms, investors and banks respectively:

$$\prod_G \hat{\Psi}_{0,G}\left(\hat{K},X\right),\ \prod_G \bar{\Psi}_{0,G}\left(\bar{K},X\right),\ \prod_G \Psi_{0,G}\left(K,X\right) \tag{7}$$

4 Basis and Fiber Variables

4.1 Collective States as Collections of Basis and Fiber Variables

The collective states described previously depend, for each group, on the levels of average stakes, returns and disposable capital per sector. Thus, we must determine, for each decomposition of the sector space in groups, the levels of disposable capital and returns for each type of agents for each sector for each group, and the distribution of stakes across all these sectors. Taken together, these quantities will define one collective state of the system.

Since returns and disposable capital per sector can be derived from the levels of stakes between investors, a collective state is described by a set of values of stakes for investors. We will call these variables the *basis variables*, while the remaining variables, that are functions of these basis variables in a given collective state, will be called *fiber variables*.

For a given collective state, the fiber variables are determined by the distribution of the basis variables, here the sectoral stakes. However, their expression is part of the complete description of the collective state.

[18] See Appendix 1,

4.2 Basis background fields and variables

4.2.1 Basis background fields

The organization of collective states in groups depends on the level of stakes within the system. The basis background fields are thus the fields of stakes for the groups composing the collective state:

$$\left\{\Gamma_{0,G}\left(\hat{S},X',X\right),\bar{\Gamma}_{0,G}\left(\bar{S},X',X\right)\right\}_{G}$$

4.2.2 Basis variables

A collective state is characterized by its basis variables, here the sectoral stakes of investors on the one hand, and on the other hand the sectoral stakes of banks wether in other banks or investors:

$$\begin{aligned}&\left\{\hat{S}_{E}\left(X',X\right),\hat{S}\left(X',X\right),S_{E}\left(X,X\right),S\left(X,X\right),\right.\\&\left.\hat{S}_{E}^{B}\left(X',X\right),\hat{S}^{B}\left(X',X\right),S_{E}^{B}\left(X,X\right),S^{B}\left(X,X\right),\bar{S}_{E}\left(X',X\right),\bar{S}^{B}\left(X',X\right)\right\}_{\left(X',X\right)\in S}\end{aligned} \tag{8}$$

These values close the formulas (1), (2), (4), (5), (3) for the background fields.

These sectoral stakes are governed by the return equations of investors and banks[19]. These are non-local equations that depend on the connections within each group. However, since under uncertainty, agents connect to only a finite number of neighbouring agents, these agents are organize into several and loosely-connected groups. Each collection of groups describes a certain decomposition of the sector space S. Besides, each of these groups G can be in one among several possible phase and each of these phases is defined by a distribution of stakes between agents in G:

$$\begin{aligned}&\left\{\hat{S}_{E}\left(X',X\right),\hat{S}\left(X',X\right),S_{E}\left(X,X\right),S\left(X,X\right),\right.\\&\left.\hat{S}_{E}^{B}\left(X',X\right),\hat{S}^{B}\left(X',X\right),S_{E}^{B}\left(X,X\right),S^{B}\left(X,X\right),\bar{S}_{E}\left(X',X\right),\bar{S}^{B}\left(X',X\right)\right\}_{\left(X',X\right)\in G}\end{aligned} \tag{9}$$

Thus, under the assumption of independent groups, a collective state is a set of sub-collective states, that writes in terms of stakes:

$$\begin{aligned}&\cup_{G}\left\{\hat{S}_{E}\left(X',X\right),\hat{S}\left(X',X\right),S_{E}\left(X,X\right),S\left(X,X\right),\right.\\&\left.\hat{S}_{E}^{B}\left(X',X\right),\hat{S}^{B}\left(X',X\right),S_{E}^{B}\left(X,X\right),S^{B}\left(X,X\right),\bar{S}_{E}\left(X',X\right),\bar{S}^{B}\left(X',X\right)\right\}_{\left(X',X\right)\in G}\end{aligned} \tag{10}$$

Since there is an infinite number of organization of the sector space into collections of groups $\{G\}$, and since for each organization, multiple distinct phases are possible for each group, there is an infinite number of possible collective states (10).

4.3 Fiber background fields and variables

Once the decomposition into groups and the values of sectoral stakes for this decomposition fixed, the collective states are completely determined. The remaining variables describing the state, the fiber variables, can then be derived for each collective state. These fiber variables are the sectoral background fields, i.e. the capital-average of the background fields of capital the average disposable capital per sector, the returns for firms, investors and banks per sectors. For the sake of simplicity, we only present the two first of these variables, the other ones are given in Appendix 3.

[19] See Appendix 1.

4.3.1 Fiber background fields

The fiber background fields of the system complete the description of the collective state. They consist in the collection of background fields for each group of the collective state:

$$\left\{\hat{\Psi}_{0,G}\left(\hat{K},X\right),\, \bar{\Psi}_{0,G}\left(\bar{K},X\right),\, \Psi_{0,G}\left(K,X\right)\right\}_G$$

4.3.2 Fiber variables

First fiber variable: Sectoral background fields The first fiber variable considered are, the average over the disposable capital K of $\hat{\Psi}_{0,G}\left(\hat{K},X\right)$ and $\bar{\Psi}_{0,G}\left(\bar{K},X\right)$ respectively, for each sector. These average sectoral fields are denoted $\hat{\Psi}\left(X\right)$, $\bar{\Psi}\left(X\right)$ and defined by:

$$\left|\hat{\Psi}_0\left(X\right)\right|^2 \simeq \left(\frac{\left\langle\hat{g}\right\rangle}{\hat{g}\left(X\right)}\hat{I}_{X/\left\langle X'\right\rangle}\right)^{\frac{3}{2}}\left\|\hat{\Psi}\right\|^2 \tag{11}$$

and:

$$\left|\bar{\Psi}_0\left(X\right)\right|^2 \simeq \left(\frac{\left\langle\bar{g}\right\rangle}{\bar{g}\left(X\right)}\bar{I}_{X/\left\langle X'\right\rangle}\right)^{\frac{3}{2}}\left\|\bar{\Psi}\right\|^2 \tag{12}$$

where the variables $\hat{I}_{X/\left\langle X'\right\rangle}$ and $\bar{I}_{X/\left\langle X'\right\rangle}$ measure the relative investment of investors and banks X' in investors and banks X and are written as:

$$I_{X/\left\langle X'\right\rangle} = \frac{\frac{\left\langle\hat{s}\left(X,X'\right)\right\rangle_{X'}}{1-\left\langle\hat{S}\left(X,X'\right)\right\rangle_{X'}}}{\frac{\left\langle\hat{S}\left(X,X'\right)\right\rangle}{1-\left\langle\hat{S}\left(X,X'\right)\right\rangle}} \tag{13}$$

and:

$$\bar{I}_{X/\left\langle X'\right\rangle} = \frac{\frac{\left\langle\bar{s}\left(X',X\right)\right\rangle_{X}}{1-\left\langle\bar{S}\left(X',X\right)\right\rangle_{X'}}}{\frac{\left\langle\bar{S}\left(X',X\right)\right\rangle}{1-\left\langle\bar{S}\left(X',X\right)\right\rangle}}$$

Thus, the sectoral background fields depend on the stakes, and their average over the whole sector space.

Second Fiber Variable: Disposable capital The second fiber variable considered is the disposable capital per type of agent per sector. It is the sum over the whole set of one type of agents in a given sector of their disposable capital. It comprises the agents private capital and the capital invested in them by the whole set of cross-sectoral agents.

For a given distribution of stakes and returns, and defining the average disposable capital[20] over the whole sector space for banks and investors by $\left\langle\bar{K}\right\rangle\left\|\bar{\Psi}\right\|^2$ and $\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2$, respectively, the disposable capital can be written, for banks:

$$\bar{K}_X\left|\bar{\Psi}\left(X\right)\right|^2 \simeq \left\langle\bar{K}\right\rangle\left\|\bar{\Psi}\right\|^2\left(\frac{\left\langle\bar{g}\right\rangle}{\bar{g}\left(X\right)}\bar{I}_{X/\left\langle X'\right\rangle}\right)^2 \tag{14}$$

for investors:

$$\hat{K}_X\left|\hat{\Psi}\left(X\right)\right|^2 \simeq \left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2\left(\frac{\left\langle\hat{g}\right\rangle}{\hat{g}\left(X\right)}I_{X/\left\langle X'\right\rangle}\right)^2 \tag{15}$$

[20] defined in Gosselin and Lotz (2025b).

and for firms:

$$K_X |\Psi (X)|^2 \simeq \left(1 - \left(S(X,X) \frac{\hat{K}_X \left|\hat{\Psi}(X)\right|^2}{\left(K_X \|\Psi (X)\|^2\right)_0} + S^B(X,X) \frac{\bar{K}_X \left|\bar{\Psi}(X)\right|^2}{\left(K_X \|\Psi (X)\|^2\right)_0}\right)\right) \left(K_X \|\Psi (X)\|^2\right)_0 \tag{16}$$

where $\left(K_X \|\Psi (X)\|^2\right)_0$ represents the disposable capital of firms X in the absence of investor participation[21], i.e. when $S(X,X) = 0$.

These disposable capital depend on the level of stakes, so that the description of the collective states relies, in last analysis, on the sectoral stakes.

Third Fiber Variable: Returns The firms, investors and banks returns are the third fiber variables. In a collective state, they are functions of averages stakes and some externla parameters, interest rates per sector and type of agents $r(X)$, $\hat{r}(X)$, $\bar{r}(X)$ and firms productivity per sector $f_1(X)$

Firms returns The firms' excess return, $R_{exc}(X)$ under decreasing returns to scale, is given by:

$$R_{exc}(X) = \frac{1}{2} f_a - \frac{1}{2} f_b \left(\frac{C_0 + \bar{r}}{f_1(X)}\right)^{\frac{1}{r}}$$

where f_a and f_b have been computed in Gosselin and Lotz (2025b), C_0 is a fixed cost, and $\bar{r}$ average interest rate faced by firms.

Investors' returns Investors' returns per sector depend on three coefficients. First, the investors' estimation of risks in investors stakes, denoted z and defined by[22]:

$$z = \left\langle \hat{S}_E(X', X) \right\rangle$$

Second, the relative strength of banks' investments in investors, denoted[23] x:

$$x = \left\langle \bar{S}_E(X', X) \right\rangle$$

Third, the relative average number of banks compared to investors, denoted P, defined by:

$$P = \frac{\left\|\bar{\Psi}_0\right\|^2}{\left\|\hat{\Psi}_0\right\|^2}$$

The higher the ratio P, the higher the importance of the banking system and its impact on intermediation. The higher the uncertainty of banks regarding investors, the lower are x and the impact of banks on investors. Under high uncertainty, banks reduce both participations and loans. Conversely, the higher the coefficient z, the higher is the intermediation among investors, and thereby the higher the impact of banks on investors.

[21] See Appendix 1.

[22] The link between z and z_0, the value of $\left\langle \hat{S}_E(X', X) \right\rangle$ under the constraint $\langle f(X) \rangle - \langle \bar{r}(X) \rangle = 0$, is given in Appendix 5.4.1 of Gosselin and Lotz (2025). In first approximation, we can consider $z = z_0$.

[23] To the first approximation we identify $x = \left\langle \bar{S}_E(X', X) \right\rangle = \left\langle \bar{S}_E \right\rangle_0$ with $\left\langle \bar{S}_E \right\rangle_0$ defined in section 12.1.3.. First order corrections to this approximation are defined in Appendix 6.5.2.

Depending on the value of P, several solutions exist for investors'returns. We refer to Gosselin and Lotz (2025b) for a complete description and focus on the main case.

When the relative average number of banks is below a certain threshold:

$$P < Th$$

the investors' return per sector have two solutions, as in Gosselin and Lotz (2025). These two solutions for investors' excess returns depend on firms' excess returns per sector as well as the average returns in the system. They are given by[24]:

$$\hat{R}_{exc}^{H}(X) = \frac{\hat{R}_{exc}^{H,0}(X)}{2} + \sqrt{\left(\frac{\hat{R}_{exc}^{H,0}(X)}{2}\right)^2 - \frac{1-2z}{1-z}\frac{\langle\bar{r}(X)\rangle}{|a(z,P)|}\hat{R}_{exc}^{L,0}(X)} \tag{17}$$

and:

$$\hat{R}_{exc}^{L}(X) = \frac{\hat{R}_{exc}^{H,0}(X)}{2} - \sqrt{\left(\frac{\hat{R}_{exc}^{H,0}(X)}{2}\right)^2 - \frac{1-2z}{1-z}\frac{\langle\bar{r}(X)\rangle}{|a(z,P)|}\hat{R}_{exc}^{L,0}(X)} \tag{18}$$

with:

$$\hat{R}_{exc}^{H,0}(X) = \frac{\frac{1-2z}{1-z}\langle\bar{r}(X)\rangle + z\left(b(z,L)\left\langle\hat{R}_{exc}(X')\right\rangle + \frac{1}{2}\frac{1-2z}{(z-1)^2}R_{exc}(X)\right)}{2|a(z,P)|} \tag{19}$$

and:

$$\hat{R}_{exc}^{L,0}(X) = z\left\langle\hat{R}_{exc}(X')\right\rangle + \frac{(1-z)}{2}R_{exc}(X) \tag{20}$$

Here, the functions $a(z,P)$ and $b(z,L)$ are defined in Gosselin and Lotz (2025b).

Banks' returns For banks, there are two solutions that have a form similar to those of investors:

$$\bar{R}_{exc}(X) = \frac{\bar{R}_{exc}^{H,0}(X)}{2} \pm \sqrt{\left(\left(\frac{\bar{R}_{exc}^{H,0}(X)}{2}\right)^2 - (\kappa+2)(\kappa+1)\frac{1-x}{x}\bar{R}_{exc}^{L,0}(X)\right)^2}$$

with:

$$\bar{R}_{exc}^{H,0}(X) = (\kappa+2)(\kappa+1)\left(\frac{1-x}{2x}\langle\bar{r}(X')\rangle + \frac{(\langle f(X)\rangle - \langle\bar{r}(X')\rangle)}{4} + \left(\left\langle\hat{f}(X')\right\rangle - \langle\bar{r}(X')\rangle\right)x\right)$$

and:

$$\begin{aligned}\bar{R}_{exc}^{L,0}(X) &= \left\langle\bar{S}_E(X',X)\right\rangle_{X'}\left\langle\overline{DF}(X')\right\rangle\left(\left\langle\bar{f}(X')\right\rangle - \bar{r}\right)\\ &+ \left\langle\hat{S}_E^B(X',X)\right\rangle_{X'}\left\langle\widehat{DF}(X')\right\rangle\left(\left\langle\hat{f}(X')\right\rangle - \bar{r}\right) + S_E^B(X,X)\left(f_1(X)_{dr} - \bar{r}\right)\end{aligned}$$

4.4 Defaults in Collective States

Defaults states do not emerge, as such, in collective states. They can be specifically accounted for, as part of the equilibrium, or they the resolution can reveal sub-collective states presenting negative returns in some sectors. In both cases, one must go back to the resolution and assume ex-ante one or several defaulting firms in some sectors. The default state can then be built recursively from

[24] See Appendix 7 in Gosselin and Lotz (2025) for the derivation.

this initial impact, to take into account the fact that, once the initial default has propagated, the returns in each sector will be deviations from what would have been the returns in a non-default scenario.

As such, default states are modifications of non-default states. Three relations characterize the default states[25].

4.4.1 Condition for the propagation of defaults

From firms to investors and banks For defaults to propagate from firms to investors and banks, firms' excess returns must be below a threshold D_{Th}, so that:

$$R(X) < D_{Th}$$

This threshold is a function of the stakes of the group[26], and the default condition can be written in terms of stakes which will be useful later on to account for defaults in transitions.

From investors and banks to other investors and banks Given an initial default among investors and banks in some sectors, the condition for default to propagate to other investors and banks can be written as:

$$\begin{aligned} &\mathbf{H}\left(\left\langle\left(\hat{S}_L\left(X',X\right)\right)\right\rangle_{X'} + S_L\left(X,X\right)\right) \qquad (21)\\ &+\mathbf{G}\left(\left\langle\bar{S}_L\left(X',X\right)\right\rangle_{X'} + \left\langle\hat{S}_L^B\left(X',X\right)\right\rangle_{X'} + S_L^B\left(X,X\right)\right) > 2\left\langle\bar{f}\right\rangle \end{aligned}$$

where $\mathbf{H}$ and $\mathbf{G}$ are functions of stakes, uncertainty and capital ratios under the non-default scenario[27].

4.4.2 Losses in a state of realized default

Ultimately, the average loss incured by the remaining banks and investors relatively to a non-default scenario can be obtained as a shift in agents' returns. Denoting $d\hat{f}$, $d\bar{f}$ and df are the average losses incured by the remaining investors, banks, and firms, for each defaulting sector, and μ is the fraction of investors impacted by the default[28], we have the following expressions.

Losses incured by investors

$$\hat{f}(X) \to \hat{f}(X) - \mu d\hat{f} \qquad (22)$$

Losses incured by banks

$$\bar{f}(X) \to \bar{f}(X) - \mu d\bar{f} \qquad (23)$$

Losses incured by firms

$$f(X) \to f(X) - \mu df \qquad (24)$$

These three relations are structural and characterize a group in a given state defined by its sectoral stakes. They are relevant in this work, since these relations will impact the possibility for a state to experience a transition towards a default state.

[25] Details are given in Appendix 1.
[26] The formula for the threshold is given in Appendix 1.
[27] Their formula are in Appendix 1.
[28] The detailed formulas are in Appendix 1.

5 Probabilities

5.1 Probabilty of Sub-Collective States

Minimizing the action functional of the system for any group G_a yields several solutions. Each are local maxima of probability and as such define a sub-collective state. However, these local maxima and the collective state they define are not equally probable.

Any sub-collective state defined by $\left\{\Gamma_0, \bar{\Gamma}_0, \Psi_0, \hat{\Psi}_0, \bar{\Psi}_0\right\}$ has an associated probability given by[29]:

$$\exp\left(-S\left(\Gamma_0\right) - S\left(\bar{\Gamma}_0\right) - S\left(\Psi_0\right) - S\left(\hat{\Psi}_0\right) - S\left(\bar{\Psi}_0\right)\right) \tag{25}$$

where the action functionals $S\left(\Gamma_0\right)$,.... involved in (25) are estimated by:

$$S\left(\Gamma_0\right) \simeq -2\sigma_{\hat{K}}^2 4 \int \left|\Gamma_0\left(\hat{S}_\eta^{(T)}, X', X\right)\right|^2 d\left(\hat{S}_\eta^{(T)}, X', X\right) \tag{26}$$

$$S\left(\bar{\Gamma}_0\right) = -3\sigma_{\hat{K}}^2 \int_0^\dagger \left|\Gamma_0\left(\bar{S}^{(T)}, X', X\right)\right|^2 d\left(\bar{S}^{(T)}, X', X\right) \tag{27}$$

$$\bar{S}\left(\bar{\Psi}_0\right) \simeq \int\left(\left(\frac{\bar{g}^2\left(X\right)}{2\sigma_{\bar{K}}^2} + \frac{\bar{g}\left(X\right)}{2\bar{K}}\right)\left|\bar{\Psi}\left(X\right)\right|^2\right) dX \tag{28}$$

$$\hat{S}\left(\hat{\Psi}_0\right) = \int\left(\left(\frac{\hat{g}^2\left(X\right)}{2\sigma_{\hat{K}}^2} + \frac{\hat{g}\left(X\right)}{2\hat{K}}\right)\left|\hat{\Psi}\left(X\right)\right|^2\right) dX \tag{29}$$

When we will consider the dynamic deviations of the fields around their background fields, these action functionals will be expanded around the background fields.

5.2 Probability of Collective States

Any collective state refers to a decomposition of the sector space in groups $\{G\}$. The background field of this collective state is, in first approximation, the product of the background fields (6) and (7) of these groups.

Since the background fields of these group are independent, the probability of the collective state will be the product of the probability (25) of these sub-collective states:

$$\prod_G \exp\left(-S\left(\Gamma_{0,G}\right) - S\left(\bar{\Gamma}_{0,G}\right) - S\left(\Psi_{0,G}\right) - S\left(\hat{\Psi}_{0,G}\right) - S\left(\bar{\Psi}_{0,G}\right)\right) \tag{30}$$

that is:

$$\exp\left(-\sum_G\left(S\left(\Gamma_{0,G}\right) + S\left(\bar{\Gamma}_{0,G}\right) + S\left(\Psi_{0,G}\right) + S\left(\hat{\Psi}_{0,G}\right) + S\left(\bar{\Psi}_{0,G}\right)\right)\right) \tag{31}$$

where the index G in (31) runs over the groups involved in the collective state. Formula (31) reveals that the action functional for the collective state is actually the sum:

$$\sum_G\left(S\left(\Gamma_{0,G}\right) + S\left(\bar{\Gamma}_{0,G}\right) + S\left(\Psi_{0,G}\right) + S\left(\hat{\Psi}_{0,G}\right) + S\left(\bar{\Psi}_{0,G}\right)\right)$$

[29]The probability of the collective state is, up to a normalization factor:

$$\exp\left(-S\left(\Gamma_0\right) - S\left(\bar{\Gamma}_0\right) - \bar{S}\left(\bar{\Psi}_0\right) - \hat{S}\left(\hat{\Psi}_0\right)\right)$$

Note that, in the context of Gosselin and Lotz (2025a,b), the fields $\bar{\Psi}_0$ and $\hat{\Psi}_0$ were determined by the stakes, so that the probability was proportional to:

$$\exp\left(-S\left(\Gamma_0\right) - S\left(\bar{\Gamma}_0\right)\right)$$

6 Fluctuations Around Collective States

We showed[30] that the set of defaulting agents in a group ultimately results from a cascade of defaults among firms, investors and banks. This suggests that shocks could trigger transitions in sub-collective states from one phase to an other and that some underlying dynamics exist between these phases. The possibility of transitions between these phases should be revealed by studying the instabilities in fluctuations of phases.

To study fluctuations around static collective states we must therefore study the phase dynamics of its sub-collective states. To do so we must transform their static return equations into dynamic ones. By introducing a time parameter, we can analyse the deviations of any group around its actual static phase, and in turn, the stability of this phase. This analysis will include the range of stability of fluctuations.

6.1 Fluctuations in Basis Variables

6.1.1 Stability: First-order dynamics

The stability of any collective state can be studied through a linear dynamic system of first-order fluctuations in the basis variables, here the stakes, around the collective state static equilibrium.

Defining $S^T(X,\theta-1)$ the total stake of capital invested in firms by:

$$S^T(X,\theta-1) = S(X,\theta-1) + S_E^B(X,\theta-1) + S_L^B(X,\theta-1) \tag{32}$$

the linear dynamic system[31] will involve the set of fluctuations variables:

$$\left(\delta\bar{S}(X',X,\theta), \delta\hat{S}(X',X,\theta), \delta S^T(X,\theta-1)\right)$$

and writes[32]:

$$\delta\mathbf{S}(X',X,\theta) = M(X)\,\delta\mathbf{S}(X',X,\theta-1) + \int dX' N(X,X')\,\delta\mathbf{S}(X'',X,\theta-1)$$

where:

$$\delta\mathbf{S}(X',X,\theta) = \begin{pmatrix} \delta\bar{S}(X',X,\theta) \\ \delta\hat{S}(X',X,\theta) \\ \delta S^T(X,,\theta-1) \end{pmatrix}$$

where the matrix $M(X)$ captures the dynamics for average stakes, and the matrix $N(X,X')$ measures the propagation of fluctuations within the system[33]. This system allows the computation of the system's eigenvalues, and the analysis of stability of the collective states[34].

6.1.2 Range of stability: second-order dynamics

The first-order dynamics determined the conditions of stability for collective states. However, even a stable collective state can be modified by sufficiently large fluctuations. To study the range of stability of a collective state, i.e. the magnitude of fluctuations that do not modify the equilibrium, we must consider a second-order dynamical system.

[30]Our analysis of default states is given in Gosselin and Lotz (2025a,b). This set-up is recalled here..
[31]The derivation is given in Gosselin and Lotz (2025b)
[32]See Appendix 1.
[33]The various matrices are defined in Appendix 4.
[34]The results were presented in Gosselin and Lotz (2025a,b).

Quadratic corrections To study the stability range of equilibria, we complete this system by adding second-order terms in fluctuations, so that the system is written in matricial form as:

$$\begin{aligned}\delta\mathbf{S}\left(X',X,\theta\right) &= M\left(X\right)\delta\mathbf{S}\left(X',X,\theta-1\right)+\int dX'N\left(X,X'\right)\delta\mathbf{S}\left(X',X,\theta-1\right)\\ &+\left(\delta\mathbf{S}\left(X',X,\theta-1\right)\right)^{t}\begin{bmatrix}\bar{M}\\ \hat{M}\\ M^{T}\end{bmatrix}\delta\mathbf{S}\left(X',X,\theta-1\right)\end{aligned}$$

The non-linear squared contribution corrects the linear approximations and allows to define approximatively the convergence zone of the collective states.

Such a system can be studied at two levels. At the average level, its eigenvalues determine the overall stability properties of a group. At the sectoral level, the fluctuations of several sectors in a given group can be considered and their local instability may induce modification of the system globally.

Range of stability The range of stability of the collective states can be evaluated from the non-linear part of the dynamics. A system is stable when its fluctuations are below a certain range. If we consider the average of the squared eigenvalues of the first-order system, denoted $\langle\lambda^2\rangle$, and a vector of fluctuations of stakes, written as:

$$\delta\mathbf{S}=\begin{pmatrix}\delta\bar{S}\left(X',X\right)\\ \delta\hat{S}\left(X',X\right)\\ \delta S^{T}\left(X\right)\end{pmatrix}$$

this system will remain stable as long as the magnitudes of these fluctuations satisfy the condition:

$$\left(1-\langle\lambda^2\rangle\right)\left(\delta\mathbf{S}\right)^2>\left(\delta\mathbf{S}^t\bar{M}\delta\mathbf{S}\right)^2+\left(\delta\mathbf{S}^t\hat{M}\delta\mathbf{S}\right)^2+\left(\delta\mathbf{S}^tM^T\delta\mathbf{S}\right)^2 \tag{33}$$

The detailed formulas for this condition (33) is given in Appendix 5.

Particular cases As a first approximation, we can consider two particular cases separately. First, the stability range for fluctuations in banks' stakes, and second, for fluctuations in investors stakes.

Banks' stakes fluctuations Fluctuations in banks stakes will remain stable if their magnitudes satisfy the following condition:

$$\begin{aligned}&\left(1-\langle\lambda^2\rangle\right)\left(\left(\delta\bar{S}\left(X',X\right)\right)^2+\left(\delta S^{T}\left(X\right)\right)^2\right)\\ >&\left(\left\{S_1^B\left(X,X\right)\left(\frac{\bar{h}\left(X\right)}{2}A\left(\delta S^{T}\left(X\right)\right)^2+B\left(\delta\bar{S}\left(X',X\right)\right)^2+\delta S^{T}\left(X\right)C\delta\bar{S}\left(X',X\right)\right)\right\}^2\right.\\ &\left.+\left\{S_1\left(X,X\right)\left(\frac{\hat{h}\left(X\right)D\left(\delta S^{T}\left(X\right)\right)^2}{2}+L\left(\delta\bar{S}\left(X',X\right)\right)^2\right)\right\}^2\right)\\ &\times\left\{\left(\varepsilon_1\left(X\right)\right)^3\left(h_1^B\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_1^B\left(X\right)\right\rangle\right)+\frac{\hat{h}\left(X\right)}{2}S\left(X\right)\right)\right\}^2\end{aligned}$$

where the coefficients A, B, C, D, and L are defined in Appendix 4 and $\varepsilon_i\left(X\right)=\frac{\partial f(X)}{\partial S^{T}(X,\theta-i)}$ representnt les elasticites des retours en fonction des stakes passees. In first approximation:

$$1-\langle\lambda^2\rangle\simeq1-\left(S^{T}\left(X\right)\frac{\partial_{f(X)}K_X\left|\Psi\left(X\right)\right|^2}{K_X\left|\Psi\left(X\right)\right|^2}\varepsilon_2\left(X\right)\right)^2 \tag{34}$$

This condition is structural: it provides an upper bound (34) under which fluctuations remain stable. This bound depends on the total level of stakes $S^T(X)$ invested in X firms and on the elasticities $\varepsilon_i(X)$. It decreases with $S^T(X)$ and $\varepsilon_2(X)$, the larger the shares received by firms in the sector, and the more reactive returns are to variations in these shares, the smaller the stability domain of fluctuations.

Investors' stakes fluctuations Fluctuations in investors stakes will remain stable if their magnitudes satisfy the following condition:

$$\begin{aligned}&\left(1-\langle\lambda^2\rangle\right)\frac{\left(\delta\bar{S}(X',X)\right)^2+\left(\delta S^T(X)\right)^2}{\left(\varepsilon_1(X)\right)^2}\\ > &\left\{\frac{8S_1(X,X)\frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)}\left(\delta\hat{S}(X)\right)^2}{\left(\hat{f}(X,\theta-1)\right)^2}-\frac{\hat{h}(X',X)\left(\frac{\hat{h}(X)}{2}S(X)-\frac{1}{f(X)+C_0}\right)\left(\varepsilon_2(X)\delta S(X)\right)^2}{\hat{f}(X,\theta-1)}\right\}^2\end{aligned}$$

These two cases show that the range of stability increases with risk perception. It increases with $\hat{h}(X',X)$ and decreases with $\varepsilon_1(X)$, $\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}$, $\frac{\partial f(X)}{\partial S^T(X,\theta-2)}$, and $\varepsilon_2(X)$, so that the higher the averages and the elasticities of both returns and capital with respect to stakes, the lower the range of stability.

6.2 Fluctuations in Fiber Variables

In all of the above, we only considered the fluctuations in basis variables, i.e. the stakes. However, fluctuations in fiber variables can also impact the whole system. Thus, we must now include fluctuations in fiber variables, considered as independent variables. This will expand the previous dynamical system to a 6 dimensional system.

6.2.1 Stability: first-order dynamics

As for basis variables, the stability of static solutions is studied by a first order dynamical system, in which fluctuations in fiber variables are accounted for. This leads to the modified first-order dynamical system:

$$X(\theta)=MX(\theta-1)$$

where:

$$X(\theta)=\left(\delta\bar{S}(X,\theta),\delta\hat{S}(X,\theta),\delta S^T(X,,\theta-1),\bar{\Delta},\hat{\Delta},\Delta\right)^t$$

with:

$$\Delta_{\bar{K}_X}=\frac{\delta\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}$$

$$\Delta_{\hat{K}_X}=\frac{\delta\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}$$

and:

$$\Delta_{K_X}=\frac{\delta K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}$$

the relative variation of disposable capital for banks, investors and firms respectively. The matrix M is defined in Appendix 5. This dynamical system is in general stable[35].

6.2.2 Range of stability: second-order dynamics

Quadratic corrections As for the basis states, we consider the dynamics including quadratic corrections for the whole system of basis and fiber variables:

$$X(\theta) = MX(\theta - 1) + X^t(\theta - 1)\mathbf{M}X(\theta - 1)$$

with:

$$\mathbf{M} = \left(\bar{M}, \hat{M}, M^T, M_{\bar{K}}, M_{\hat{K}}, M_K\right)^t \tag{35}$$

The matrices are given in Appendix 5.

Range of stability The estimation of the range of stability is similar to that for basis states, expanded to account for fiber variables fluctuations:

$$\left(1 - \langle\lambda^2\rangle\right)(\delta X)^2 > \sum_i \left(\delta X^t \mathbf{M}_i \delta X\right)^2 \tag{36}$$

where the matrices $\mathbf{M}_i$ are the components of $\mathbf{M}$ defined in (35).

Particular cases As a first approximation, we will consider successively fluctuations in banks' variables only, then fluctuations in investors' variables.

Fluctuations in banks' fiber variables Considering banks only, the stability condition involving fiber fluctuations can be approximated by:

$$\begin{aligned}&\left(1 - \langle\lambda^2\rangle\right)\\ > \quad & \frac{A\left(\delta\bar{S}(X)\,\delta S^T(X)\right)^2 + \left(B_1\left(\delta S^T(X)\right)^2 + B_2\delta S^T(X)\left(\bar{\Delta} - \Delta\right)\right)^2 + C\left(\left(\delta S^T(X)\right)^2\right)^2}{(\delta\mathbf{S})^2 + \left(\delta\mathbf{K}_X\left|\mathbf{\Psi}(X)\right|^2\right)^2}\end{aligned}$$

where the matrices A, B_1, B_2, and C given in Appendix 5, and:

$$\left(\delta\mathbf{K}_X\left|\mathbf{\Psi}(X)\right|^2\right)^2 = \left(\delta\bar{K}_X\left|\bar{\Psi}(X)\right|^2\right)^2 + \left(\delta\hat{K}_X\left|\hat{\Psi}(X)\right|^2\right)^2 + \left(\delta K_X\left|\Psi(X)\right|^2\right)^2$$

is the total squared variation of disposable capital.

Fluctuations in investors' fiber variables When investors are considered in isolation, the domain is approximated by:

$$\begin{aligned}&\left(1 - \langle\lambda^2\rangle\right)\\ > \quad & \frac{A\left(\left(\delta S(X)\right)^2\right)^2 + \left(B_1\delta S(X) + B_2\left(\hat{\Delta} - \Delta\right)\right)^2}{(\delta\mathbf{S})^2 + \left(\delta\mathbf{K}_X\left|\mathbf{\Psi}(X)\right|^2\right)^2}\end{aligned}$$

[35] See Appendix 5.

where the matrices A, B_1, B_2 given in Appendix 5 and the squared variation:

$$\begin{aligned}&(\delta\mathbf{S})^2+\left(\delta\mathbf{K}_X\left|\mathbf{\Psi}(X)\right|^2\right)^2\\=&\left(\left(\delta\hat{S}(X',X)\right)^2+\left(\delta S^T(X)\right)^2\right)+\left(\delta\hat{K}_X\left|\hat{\Psi}(X)\right|^2\right)^2+\left(\delta K_X\left|\Psi(X)\right|^2\right)^2\end{aligned}$$

is reduced to investors and firms.

7 Description of Collective States with Interacting Groups

Introducing a distinction between basis and fiber variables fluctuations amongs to doubling the variables of the system. We will thus introduce a more compact notation to describe the collective states, notation that will also allow us for interactions between groups.

7.1 General Description

Each collective state is described by a collection of background fields :(1), (2), (4), (5), (3):

$$\left\{\left|\Psi(K,X)\right|^2,\left|\bar{\Psi}\left(\bar{K},X\right)\right|^2,\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2,\Gamma_{0,X',X}\left(\hat{S}^{(T)}\right),\Gamma_{0,X',X}\left(\bar{S}^{(T)}\right)\right\}_{(X',X)}$$

with the set of average sectoral stakes of the form (8):

$$\left\{\bar{S}_\eta(X',X),S_\eta^B(X,X),\hat{S}_\eta^B(X',X),\hat{S}_\eta(X',X),S_\eta(X,X)\right\}\tag{37}$$

where $\eta=E,L$ describes shares and loans, respectively.

The fiber variables, here returns and disposable capital, can be expressed as functions of these basis variables. These levels of stakes of a collective state are not arbitrary, they are the solutions of the various minimization equations of the system, also called *saddle point stakes*, and will be denoted, for later purpose, with a subscript to distinguish them from any arbitrary levels of stakes taken by the fluctuations around the saddle point stakes. Thus, a description of a collective state is given ultimately by a collection of basis variables:

$$\begin{aligned}&\left\{\underline{\bar{S}_\eta}(X',X),\underline{S_\eta^B}(X,X),\underline{\hat{S}_\eta}^B(X',X),\underline{\hat{S}_\eta}(X',X),\underline{S_\eta}(X,X)\right\}\\\equiv&\left[\underline{\bar{S}_\eta},\underline{S_\eta^B},\underline{\hat{S}_\eta}^B,\underline{\hat{S}_\eta},\underline{S_\eta}\right]\end{aligned}$$

When there will be no ambiguity, we will denote this collection:

$$[\underline{\mathbf{S}}]$$

In the sequel, to keep track that a collective state refers to some realization of the fields, and not only on a set of stakes we will write a collective state as:

$$\prod_{(X',X)}\left|\underline{\bar{S}_\eta}(X',X),\underline{S_\eta^B}(X,X),\underline{\hat{S}_\eta}^B(X',X),\underline{\hat{S}_\eta}(X',X),\underline{S_\eta}(X,X)\right\rangle$$

This notation will be useful to include interactions between groups.

This collective state can also be written alternately:

$$\begin{aligned}&\left|\left(\underline{\bar{S}_\eta}(X',X),\underline{S_\eta^B}(X,X),\underline{\hat{S}_\eta}^B(X',X),\underline{\hat{S}_\eta}(X',X),\underline{S_\eta}(X,X)\right)_{(X',X)}\right\rangle\\\equiv&\left|[\underline{\mathbf{S}}]\right\rangle\end{aligned}$$

This notation will be modified when needed to adjust the specificities of the model[36].

7.2 Decomposition in sub-collective states

When a collective state is decomposed of several interacting group, the notation has to be modified to account for their interactions. We develop the notations by first describing a single sub-collective state associated to one group, then considering multiple independent groups, and then the full case of multiple interacting groups.

7.2.1 Single Group

For a sub-collective, we use the same notation, restricted to the given group. A sub-collective state will therefore be denoted:

$$\left|\left(\underline{\bar{S}_{\eta}}(X',X),\underline{S_{\eta}^{B}}(X,X),\underline{\hat{S}_{\eta}}^{B}(X',X),\underline{\hat{S}_{\eta}}(X',X),\underline{S_{\eta}}(X,X)\right)_{(X'\in G,X\in G)}\right\rangle$$

or, more compactly:

$$|[\underline{\mathbf{S}},G]\rangle \tag{38}$$

7.2.2 Multiple Independent Groups

Since collective states organize into groups that are independent, with their particular connections, we must consider states that are decomposed into products of several states:

$$\begin{aligned}&\prod_{G}\left|\left(\underline{\bar{S}_{\eta}}(X',X),\underline{S_{\eta}^{B}}(X,X),\underline{\hat{S}_{\eta}}^{B}(X',X),\underline{\hat{S}_{\eta}}(X',X),\underline{S_{\eta}}(X,X)\right)_{(X'\in G,X\in G)}\right\rangle\\ \equiv\ &\prod_{G}|[\underline{\mathbf{S}},G]\rangle\end{aligned} \tag{39}$$

corresponding to any arbitrary decomposition in groups G of sectors, to describe the collection of product of background fields 6) and (7):

$$\left\{\prod_{G}\Gamma_{0,G}\left(\hat{S},X',X\right),\prod_{G}\bar{\Gamma}_{0,G}\left(\bar{S},X',X\right),\prod_{G}\hat{\Psi}_{0,G}\left(\hat{K},X\right),\prod_{G}\bar{\Psi}_{0,G}\left(\bar{K},X\right),\prod_{G}\Psi_{0,G}\left(K,X\right)\right\}$$

[36] For later purpose, when only investors and firms will be considered, we will write a collective state as:

$$\prod_{(X',X)}\left|\underline{S_{\eta}}(X,X),\underline{\hat{S}_{\eta}}(X',X)\right\rangle$$

or alternately:

$$\left|\left(\underline{S_{\eta}}(X,X),\underline{\hat{S}_{\eta}}(X',X)\right)_{(X',X)}\right\rangle$$

Such a state describing the distribution of stakes for all sectors:

$$\left\{\underline{S_{E}}\left(\hat{X},\hat{X}\right),\underline{\hat{S}_{E}}\left(\hat{X}',\hat{X}\right),\underline{S_{L}}\left(\hat{X},\hat{X}\right),\underline{\hat{S}_{L}}\left(\hat{X}',\hat{X}\right)\right\}$$

7.2.3 Multiple Interacting Groups

Until now, our notation has only described independent groups. However, in a collective state, groups may be interconnected To account for these connections, we must extend our notation. To do so, we denote the collection of bilateral stakes between groups i and j by $\delta S_{G_i,G_j}$ and we write the states including these bilateral stakes as:

$$\overbrace{\prod_{G}\prod_{((X',X),(X',X))\in G}\left|\underline{\bar{S}_{\eta}}(X',X),\underline{S_{\eta}^{B}}(X,X),\underline{\hat{S}_{\eta}}^{B}(X',X),\underline{\hat{S}_{\eta}}(X',X),\underline{S_{\eta}}(X,X)\right\rangle}^{S_{G_i,G_j}} \tag{40}$$

or, in a more compact form:

$$\overbrace{\prod_{G}|[\underline{\mathbf{S}},G]\rangle}^{S_{G_i,G_j}} \tag{41}$$

When groups are organized into independent clusters of several interacting groups, to each cluster corresponds a *cluster state* of the form (41), and the full collective state is the product of these cluster states, written as:

$$\prod\overbrace{\prod_{G}|[\underline{\mathbf{S}},G]\rangle}^{S_{G_i,G_j}} \tag{42}$$

where the first product $\prod$ runs over the collection of independent clusters.

Towards Fully Dynamic Collective States

In the description above, collective states were considered static to reflect that, as aggregates, their time scale is fundamentally larger than the microeconomic time scale. However, the emergence of defaults and the stability of collective states are inherently dynamic processes. Thus, our framework should be extended to account for these dynamics.

To do so, we must introduce the concepts and assumptions needed to analyse collective states as dynamic objects, along with their expected patterns of transition, before summarizing the field theory to be developped.

8 What dynamic collective states should account for

To analyze the dynamics of collective states, we must consider them as collections of interacting sub-collective states and develop the notion of deviation states, which will be the basic necessary dynamic objects, and show how these deviation states can be formalized within a field-theoretic framework, in which each static sub-collective state is associated with its distinct field.

8.1 Dynamic sub-collective states

8.1.1 Sub-collective states

Transitions between collective states arise from fluctuations within sub-collective states. Even loose interactions between groups propagate these fluctuations[37]: for two groups with different stability

[37]See Gosselin and Lotz (2025a,b).

ranges, fluctuations originating in one group - although internally stable - may push another group beyond its own stability range, thereby potentially initiating a sequence of transitions affecting this group and others.

Thus, the transitions between collective states arise from their modifications and from the propagation of transitions among groups. They therefore must be studied at the level of sub-collective states. To simplify our analysis, we will assume that groups are interacting but loosely connected. For each sub-collective state, interactions induce fluctuations around their phase. Besides, exogenous modifications in basis variables may also induce shifts in equilibrium by the propagation of these modifications to the whole sub-collective state.

8.1.2 Interacting groups

Among these independent groups, interactions materialize through the stakes taken by investors of one group in investors of other groups, what we should call inter-group investors' stakes. This implies assuming some distributions of inter-group stakes. To do so, we must describe collective states as a set of sub-collective states including some additional stakes between groups. This describes collective states as larger structures than the collection of sub-collective states.

8.2 Deviation states

Deviation states describe the possible fluctuations around collective states, or sub-collective states when only a subset of sectors is considered. As with static collective states, deviation states decompose into *basis* and *fiber deviation states*, which may both experience transitions.

8.2.1 Basis deviation states

A collective state is a set of groups, each described by their *basis variables* - in our model investors' and banks' stakes - gathered in a *basis collective state*. This basis collective state determines the types of fiber variables of the model, which are, in our model, returns, capital, and sectoral background fields.

To introduce dynamic deviations around such a basis collective state, we must introduce the notion of *basis deviations variables*, that encompasses all the basis variables of the system, $\mathbf{S}$, i.e. the stakes, whose values differ from their equilibrium values, denoted $\underline{\mathbf{S}}$. We define a *basis deviation state* for a given group G a sub-collective state where the variables have been extended to account for $\mathbf{S}$ and $\underline{\mathbf{S}}$.

8.2.2 Fiber deviation states

In a static framework, the basis collective states determine the values of the fiber variables of the model. However, in a dynamical context, any deviation in basis variables from their equilibrium values generates a multiplicity of possible values for the fiber variables. To account for this multiplicity and fluctuations in fiber variables themselves, a dynamic framework should treat fiber variables as autonomous.

In a dynamic context, for each set of basis deviation variables of a group G, the fiber variables of the model can asume multiple values, each describing a *fiber equilibrium state*.

Moreover, as for basis variables, some *fiber deviation states* arise around these fiber equilibrium states. they will be described by a set of *fiber deviation variables*.

The dynamic description of the fiber will thus both depend on the fiber equilibrium states and the associated fiber deviation states. The dynamics thus acts on both the basis and the fiber states, with changes and transitions in each type, and transitions in each type being mutually dependent.

8.2.3 Fibered deviation states

Each dynamic collective state should combine a basis deviation state and one among its possible associated fiber deviation state. We will denote this combination a *fibered deviation state.* There are an infinite number of fibered deviation states.

Since these two deviation states are not independent, and influence each other, a fibered deviation state is not merely the sum of the deviation ststaes that compose it, but also includes the impact of their interactions.

8.3 Deviation fields

We have considered deviation around static collective states. To model their dynamics, we have to consider these states as being themselves the realizations of some fields. Since we have introduced two deviation states, basis and fiber, we will introduce a *basis deviation field* and a *fiber deviation field*.

8.3.1 Basis deviation field

Static collective states were defined by realizations of the fields $\bar{\Gamma}_0$ and Γ_0. Similarly, we introduce some basis deviation fields around $\bar{\Gamma}_0$ and Γ_0. These fields are characterized by their arguments, the basis deviation variables, here the stakes, $\left(\bar{S}_\eta, S_\eta^B, \hat{S}_\eta^B, \hat{S}_\eta, S_\eta\right)$, and their action functionals, obtained by computing the second order expansion of the initial static action functionals $\bar{S}(\bar{\Gamma})$ and $S(\Gamma)$ around $\bar{\Gamma}_0$ and Γ_0[38].

8.3.2 Fiber deviation field

However, merely introducing the basis deviation fields is not sufficient, since they only describe dynamics in the basis states. Actually, for each arbitrary distribution of stakes, an infinite number of possible distributions of capital level and return per sector[39] may exist. We must therefore introduce some deviation fields for the fiber variables, that will account for deviation states around the multiple distributions of the fiber variables.

Our formalism will therefore be extended along two directions. First, a field theory describing the basis fluctuations, and second, a field theory for fiber fluctuations, i.e. fluctuations in capital and returns. This field theory depends explicitely on the basis fluctuations[40]. Two fields and their action functionals will model the fluctuations in capital and returns of investors and banks.

Together, these two field theories provide a description of the deviation states around a given collective state $\bar{\Gamma}_0$ and Γ_0. Since it depends explicitely on the collective state, we denote this field theory $(\Gamma, \Psi)\left[\bar{\Gamma}_0, \Gamma_0\right]$.

[38] See Appendix 1.
[39] See Gosselin and Lotz (2024),
[40] These fibers fluctuations are described by the field theory of Gosselin and Lotz (2024).

8.3.3 Fibered deviation field

The fibered deviation states should be seen as realizations of a field denoted *fibered deviation field*. However, fibered deviation states being, combinations of basis and fiber deviation states and of their interactions, any fibered deviation field can be seen as a vector of two components, the basis and fiber deviation fields.

Studying fiber deviation states will therefore amount to considering these two deviation fields and their interactions and, in the following, we will directly consider their two components, the basis et fiber deviation fields.

8.4 Dynamic patterns in basis, fiber and fibered states

Various possible dynamics and transitions may exist for each group and, by extension, for all groups within a collective state. We will describe here the types of dynamical pattern that should be expected and modeled.

8.4.1 Dynamic patterns in basis deviation states

Fluctuations in the basis variables reflect changes in agents' optimization which implies only marginal modifications in the equilibrium levels of the basis variables of the model. These fluctuations being small, they are necessarily present in any emerging transition.

However, these basis fluctuations, although marginal, can, through the interplay of group interactions, induce more radical fluctuations in the collective states and the system's equilibrium levels. Indeed, when amplified by group interactions, fluctuations can alter the phase of a sub-collective state and shift its equilibrium by driving a group beyond its stability range. Such a shift may trigger a phase transition for the group, which can in turn affect other groups and the collective state as a whole.

8.4.2 Dynamic patterns in fiber deviation states

Besides, even for relatively stable basis variables, fiber variables can experience large shifts. When levels of basis variables become arbitrary, fiber variables can assume multiple equilibria. Transitions can occur between these multiple equilibria and be triggered by inter-sector agents' interactions. These possibly sudden transitions may in turn impact the basis variables. In this two-step mechanism, fluctuations in basis variables first induce transitions in fiber variables, then in basis variables. And conversely, shifts in fiber variables can induce fluctuations in basis variables, indirectly triggering further fluctuations in fiber variables. This yields the following sequence of events: fluctuations in the fiber variables, trigger fluctuations in the basis variables, leading to a transition in the fiber variables, then in the basis variables.

Now that we have defined the dynamical patterns of both basis and fiber states, we can specify how transitions may emerge within static collective states.

8.4.3 Dynamic patterns in fibered deviation states

The interactions between the dynamics of the fiber deviation state and the dynamic of the basis deviation state is caracteristic of the dynamic of a full deviation state, that is of the fibered deviation state. The study of the dynamic collective states will therefore be the study of the fibered deviation states.

9 Modeling dynamic Collective States

Transitions between two collective states $(\bar{\Gamma}_0, \Gamma_0)$ and $(\bar{\Gamma}'_0, \Gamma'_0)$ are dynamic processes that involve both field theories $(\Gamma, \Psi)[\bar{\Gamma}_0, \Gamma_0]$ and $(\Gamma, \Psi)[\bar{\Gamma}'_0, \Gamma'_0]$. To account for these dynamic processes, we must review, define and denote the extended collective states that can emerge from including the deviations of the system's variables around their equilibrium values.

9.1 Dynamics in Basis Deviation States

As mentionned above, deviations in the basis variables of the system may emerge. Including these variations into the initial collective state yields what we will denote *basis deviation states.*

9.1.1 Basis Deviation States

A collective state is a set of groups, each described by their basis variables, gathered in a basis collective state. In the above, it was written:

$$|[\underline{\mathbf{S}}, G]\rangle = \left|\left[\underline{\bar{S}_\eta}, \underline{S_\eta^B}, \underline{\hat{S}_\eta}^B, \underline{\hat{S}_\eta}, \underline{S_\eta}, G\right]\right\rangle \tag{43}$$

This basis collective state determines the types of fiber variables of the model, which were in the above the returns $R(X)$, $\hat{R}(X)$, $\bar{R}(X)$; the capital $\hat{K}_X \left|\hat{\Psi}(X)\right|^2$, $\bar{K}_X \left|\bar{\Psi}(X)\right|^2$; and the average sectoral background fields $\left|\hat{\Psi}(X)\right|^2$, $\left|\bar{\Psi}(X)\right|^2$.

To introduce dynamic deviations around such a basis collective state, we must introduce the *basis deviations variables*, $\mathbf{S}$, that encompasses all the basis variables of the system, i.e. the stakes:

$$\mathbf{S} \equiv \left(\bar{S}_\eta, S_\eta^B, \hat{S}_\eta^B, \hat{S}_\eta, S_\eta\right)$$

The values of these basis deviation variables differ from their static equilibrium values, $\underline{\mathbf{S}}$, so that a *basis deviation state* for a given group will include these variables and will be written:

$$|[\mathbf{S}, \underline{\mathbf{S}}, G]\rangle$$

Basis deviation states will write differently depending on the composition of the initial collective state. Therefore, their notation will depend on the number of groups and the nature of their interactions.

Single group The basis deviation state associated to the group $G = \{(X', X)\}$, with the stakes $S_\eta(X, X), \hat{S}_\eta(X', X)$ as basis variables, deviating from their equilibrium values $\underline{S_\eta}(X, X), \underline{\hat{S}_\eta}(X', X)$ will be denoted[41]:

[41] In systems with two types of agents, investors and firms, we denote the sub-collective associated to a given group $G = \{(X', X)\}$ by:

$$\prod_{(X', X) \in G} \left|\underline{S_\eta}(X, X), \underline{\hat{S}_\eta}(X', X)\right\rangle$$

Similarly, we denote the deviation state associated to the group $G = \{(X', X)\}$, with stakes $S_\eta(X, X), \hat{S}_\eta(X', X)$ deviating from their equilibrium values $\underline{S_\eta}(X, X), \underline{\hat{S}_\eta}(X', X)$ as:

$$\prod_{(\hat{X}', \hat{X}) \in G} \left|S_\eta(X, X), \hat{S}_\eta(X', X), \underline{S_\eta}(X, X), \underline{\hat{S}_\eta}(X', X), \hat{K}[X]\right\rangle$$

$$\left|\left[\bar{S}_{\eta}\left(X',X\right),S_{\eta}^{B}\left(X,X\right),\hat{S}_{\eta}^{B(i)}\left(X',X\right),\hat{S}_{\eta}^{(i)}\left(X',X\right),S_{\eta}^{(i)}\left(X,X\right),\underline{\bar{S}_{\eta}},\underline{S_{\eta}}^{B},\underline{\hat{S}_{\eta}}^{B},\hat{S}_{\eta},\underline{S_{\eta}}\right]\right\rangle$$

or, using the compact notation:

$$\left|\left[\mathbf{S},\underline{\mathbf{S}},G\right]\right\rangle$$

where the parameters $\underline{\mathbf{S}}$ represent the possible of phases of the group, and $\mathbf{S}$ the values taken by the stakes when then deviate from the phase. Some of these states are unstables, and can trigger the instability of the whole collective state.

A compact expression for the stability condition of the sub-collective state characterized by $\underline{\mathbf{S}}$ can be written as:

$$C\left(\underline{\mathbf{S}},\mathbf{S}\right)<0$$

To account for dynamic collective states, we will include S as an additional variable describing these states.

Multiple independent groups For collective states composed of a collection $\{G\}$ of several independent groups, the deviation state will be denoted:

$$\prod_{G}\left|\left[\mathbf{S},\underline{\mathbf{S}},G\right]\right\rangle$$

that is, the product over the groups of the deviation states described by the set $[\mathbf{S},\underline{\mathbf{S}},G]$.

Multiple interacting groups When a collective state is composed of a collection $\{G\}$ of interacting groups, the interactions between groups must be explicitly accounted for. However, these interactions do not necessarily span the entire set of groups: heterogeneity in the intensity and structure of interactions may exist and should be taken into account. We can therefore suppose that groups organize into several independent clusters, each cluster being composed of a collection of interacting groups.

In this context, the deviation state for a cluster of independant groups would be represented by the notation:

$$\prod_{G\in Cl}\left|\left[\mathbf{S},\underline{\mathbf{S}},G\right]\right\rangle$$

However, to account for intra-cluster interactions, we use the brace:

$$\overbrace{\ldots\ldots\ldots}^{\mathbf{S}_{G\in Cl,G'\in Cl}}$$

where $\mathbf{S}_{G,G'}$ represents the cross-stakes between groups within a given cluster, so that the deviation state of this cluster of interacting groups is written:

$$\overbrace{\prod_{G\in Cl}\left|\left[\mathbf{S},\underline{\mathbf{S}},G\right]\right\rangle}^{\mathbf{S}_{G\in Cl,G'\in Cl}}$$

We will call this state a *cluster deviation state.*

Building on this notation, the deviation state describing several independent clusters will be the product of their respective cluster deviations states:

$$\prod_{Cl}\left(\overbrace{\prod_{G\in Cl}\left|\left[\mathbf{S},\underline{\mathbf{S}},G\right]\right\rangle}^{\mathbf{S}_{G\in Cl,G'\in Cl}}\right) \tag{44}$$

where $\prod_{Cl}$ denotes the product over independent clusters. Note that, when all the groups of a collective state interact, the previous notation (44) reduces to:

$$\overbrace{\prod_{G}\left|\left[\mathbf{S},\underline{\mathbf{S}},G\right]\right\rangle}^{\mathbf{s}_{G,G'}} \tag{45}$$

In the expressions above, the interactions between groups within each cluster are considered as relatively weak, which implies that the deviation stakes are smaller than the interactions within groups, so that:

$$\begin{aligned}\mathbf{S}_{G\in Cl,G'\in Cl} &< <\mathbf{S}_{G}\\ \mathbf{S}_{G\in Cl,G'\in Cl} &< <\mathbf{S}_{G'}\end{aligned}$$

9.1.2 Basis deviation fields

Deviations states are defined with respect to a reference distribution of basis variables - the stakes - $\{\underline{\mathbf{S}}\}$ for the set of groups $\{G\}$. Said differently, for each group G, a distribution $\mathbf{S}$ is a deviation from the values $\underline{\mathbf{S}}$ taken by the stakes in a given phase of the group G.

In the same way that this distribution $\underline{\mathbf{S}}$ corresponds to a given background field, the distributions of these deviations $\mathbf{S}$ should themselves be described by some fields, along with their associated field action functionals. Besides, these fields should depend on the static values $\underline{\mathbf{S}}$, but also explicitely on the groups G, to specify the sectors on which the fields are defined. We must therefore define, for each phase distribution $\underline{\mathbf{S}}$ of the group G, the deviation field:

$$\Lambda\left(\mathbf{S},\underline{\mathbf{S}},G\right)$$

The action functional for these deviation fields will be given by second order expansion of the action functionals $S(\Gamma)$ and $S(\bar{\Gamma})$ around the background fields representing a static collective state. Depending on the number of groups and their interactions, these action functionals can adopt various forms.

Single group For a single group, the action functionals for the deviation fields will be obtained by first, computing the second derivatives evaluated at their saddle point value:

$$\frac{\delta^{2}}{\delta\Gamma\delta\Gamma^{\dagger}}S\left(\Gamma\right)$$

and:

$$\frac{\delta^{2}}{\delta\bar{\Gamma}\delta\bar{\Gamma}^{\dagger}}S\left(\bar{\Gamma}\right)$$

then, by considering the second-order expansion around the background fields of $S(\Gamma)$ and $S(\bar{\Gamma})$[42].

This defines the action functional for the deviation field as:

$$\begin{aligned}S\left(\Lambda\left(\mathbf{S},\underline{\mathbf{S}},G\right)\right) &= \sum_{G}\int\Lambda^{\dagger}\left(\mathbf{S},\underline{\mathbf{S}},G\right)\left[\frac{\delta^{2}}{\delta\Gamma\delta\Gamma^{\dagger}}S\left(\Gamma\right)\right]\Lambda\left(\mathbf{S},\underline{\mathbf{S}},G\right)d\left(\mathbf{S},X',X\right)\\ &+\sum_{G}\int\Lambda^{\dagger}\left(\mathbf{S},\underline{\mathbf{S}},G\right)\left[\frac{\delta^{2}}{\delta\bar{\Gamma}\delta\bar{\Gamma}^{\dagger}}S\left(\bar{\Gamma}\right)\right]\Lambda\left(\mathbf{S},\underline{\mathbf{S}},G\right)d\left(\mathbf{S},X',X\right)\end{aligned} \tag{46}$$

[42]The formula are given by (95) and (97) in Appendix 1.

where the integrals in the action functional are restricted to G and the second order derivatives are given in Appendix 2.

For each group G in any phase $\underline{\mathbf{S}}$, the field model that we will develop will be defined by a field $\Lambda(\mathbf{S}, \underline{\mathbf{S}})$ and its action functional $S(\Lambda(\mathbf{S}, \underline{\mathbf{S}}))$.

To sum up, for any given group G, the complete field description of its basis states is given by the set of all deviation fields $\Lambda(\mathbf{S}, \underline{\mathbf{S}})$ with their action functionals, and the corresponding field model will be denoted:

$$[S(\Lambda(\mathbf{S}, \underline{\mathbf{S}}, G), G)]_{\underline{\mathbf{S}}} \tag{47}$$

Multiple independent groups For a collective state defined by a collection of independent groups $\{G\}$ with fields $\Lambda(\mathbf{S}, \underline{\mathbf{S}}, G)$, the full description of its deviation states is given by the collection, for the whole set of groups, of the fields $\Lambda(\mathbf{S}, \underline{\mathbf{S}})$ and their action functional. This action functional is obtained by summing the groups' independent action functionals:

$$\sum_G S((\Lambda(\mathbf{S}, \underline{\mathbf{S}}, G), G)) \tag{48}$$

Multiple interacting groups To model the interactions between groups and develop a complete formalism for dynamic collective states[43], we must modify (48) to account for interactions between groups and modifications in groups. To do so, we add to (48) interaction terms of the form $\delta S_{G,G'}(\Lambda(\mathbf{S}, \underline{\mathbf{S}}, G), \Lambda(\mathbf{S}, \underline{\mathbf{S}}, G))$, which yields:

$$\sum_G S(\Lambda(\mathbf{S}, \underline{\mathbf{S}}, G)) + \sum_{G,G'} \delta S_{G,G'}(\Lambda(\mathbf{S}, \underline{\mathbf{S}}, G), \Lambda(\mathbf{S}, \underline{\mathbf{S}}, G)) \tag{49}$$

The corresponding field model can be written as:

$$\prod [S(\Lambda(\mathbf{S}, \underline{\mathbf{S}}, G)), \delta S_{G,G'}(\Lambda(\mathbf{S}, \underline{\mathbf{S}}, G), \Lambda(\mathbf{S}, \underline{\mathbf{S}}, G))] \tag{50}$$

These interaction terms $\delta S_{G,G'}(\Lambda(\mathbf{S}, \underline{\mathbf{S}}, G), \Lambda(\mathbf{S}, \underline{\mathbf{S}}, G))$ will be conditioned by the transitions between sub-collective states that appear in the context of multiple interaction groups[44].

9.2 Dynamics in Fiber Deviation States

In a deviation state, the basis variable $\mathbf{S}$ differ from those defining the collective states $\underline{\mathbf{S}}$. Indeed, deviations of basis variables from their equilibrium break the set of binding optimization relations between both basis and fiber variables - stakes, capital and returns - so that for any arbitrary values of $\mathbf{S}$, there are multiple equilibrium values for the fiber variables R and K[45].

9.2.1 Fiber Deviation States

Recall that in a dynamical context, the deviations of basis variables from their equilibrium values generate a multiplicity of possible values for the fiber variables, so that a dynamic framework has to consider fiber variables as autonomous.

In our model, the fiber variables for a given level of stakes in a group G are the returns:

[43] These interactions will be investigated in the sequel of this work, while developing the field formalism for deviations stakes.

[44] The precise form of such terms will be developed in our the next paper of this series.

[45] See Gosselin and Lotz (2024).

$$\underline{\mathbf{R}}\equiv\left(R\left(X\right),\hat{R}\left(X\right),\bar{R}\left(X\right)\right)_{X\in G}$$

the capital:

$$\underline{\mathbf{K}}\equiv\left(K_{X},\hat{K}_{X},\bar{K}_{X}\right)_{X\in G}$$

and the sectoral background fields:

$$\underline{\mathbf{\Psi}}\equiv\left(\left|\Psi\left(X\right)\right|^{2}\left|\hat{\Psi}\left(X\right)\right|^{2},\left|\bar{\Psi}\left(X\right)\right|^{2}\right)_{X\in G}$$

so that, for any group G we must consider multiple equilibrium fiber states of the form:

$$|\underline{\mathbf{R}},\underline{\mathbf{\Psi}},\underline{\mathbf{K}}\rangle_{[\mathbf{S},G]}$$

where $[\mathbf{S},G]$ stands for the values of the basis variables $\mathbf{S}$ in the group G, and determine the equilibrium values of the various fiber variables of the model. However, as for basis variables, some *fiber deviations variables* $(\mathbf{R},\mathbf{\Psi},\mathbf{K})$ arise around these equilibrium.

In the sequel, we will use the notation:

$$\mathbf{F}=(\mathbf{R},\mathbf{\Psi},\mathbf{K})$$

and:

$$\underline{\mathbf{F}}=(\underline{\mathbf{R}},\underline{\mathbf{\Psi}},\underline{\mathbf{K}})$$

The fiber deviation state will therefore be written:

$$|\mathbf{F},\underline{\mathbf{F}}\rangle_{[\mathbf{S},G]}$$

The dynamics thus acts on both the basis and the fiber states, with changes and transitions in each type, and transitions in each type being mutually dependent. Unlike for basis deviation states, fiber deviation states are described solely by the set of values taken by the fiber equilibrium and deviation variables associated to a given set of basis variables of a given given group G.

9.2.2 Fiber deviation field

The deviation states of a single group are given by the equation (56), in which the fiber states for any given level of stakes $\mathbf{S}$:

$$|\mathbf{F},\underline{\mathbf{F}},G\rangle_{[\mathbf{S}]}$$

decomposes into two parts, the equilibrium part of the fiber variables and the deviation part of the fiber variables. On doit définir un champs pour chacune de ces variables.

Fiber equilibrium field First, the equilibrium fiber variables $\underline{\mathbf{F}}$ arise from a fiber field theory:

$$\left[\begin{array}{c}\hat{\Psi},\bar{\Psi}\\ \hat{S}\left(\hat{\Psi}\right),\bar{S}\left(\hat{\Psi},\bar{\Psi}\right)\end{array}\right]_{[\mathbf{S},G]}\tag{51}$$

where the realizations of the fields $\hat{\Psi},\bar{\Psi}$ and Ψ are three among the saddle point solutions of the action functional given by[46]:

$$\hat{S}_{[\mathbf{S}]}\left(\hat{\Psi}\left(K,X\right)\right)+\bar{S}_{[\mathbf{S}]}\left(\hat{\Psi}\left(K,X\right),\bar{\Psi}\left(K,X\right)\right)$$

[46]The expressions of the field action functionals $\hat{S}_{[\mathbf{S}]}\left(\hat{\Psi}\right)$ for the investors' field, and $\bar{S}_{[\mathbf{S}]}\left(\bar{\Psi},\hat{\Psi}\right)$ for banks' fields are given in Appendix 1.

where the action functionals[47] $\hat{S}_{[\mathbf{S}]}$ and $\bar{S}_{[\mathbf{S}]}$ associated to the fiber deviation fields $\bar{\Psi}$ and $\hat{\Psi}$ depend on the level of the basis variables $\mathbf{S}$.

Fiber deviation field Second, the variables $\mathbf{F}$ should themselves arise from a fiber field theory. Given an equilibrium level an equilibrium level $\underline{\mathbf{F}}$ of fiber variables, The deviations should be described by a second order expansion of the action functionals $\hat{S}\left(\hat{\Psi}\left(\hat{K},X\right)\right)$ and $\bar{S}\left(\hat{\Psi}\left(\hat{K},X\right),\bar{\Psi}\left(\bar{K},X\right)\right)$ for a fiber field:

$$\mathbf{\Lambda}\left(\mathbf{F},\underline{\mathbf{F}},G\right)$$

The action for this field is thus given by:

$$\begin{aligned} & S_F\left(\mathbf{\Lambda}\left(\mathbf{F},\underline{\mathbf{F}},G\right)\right) \\ = & \mathbf{\Lambda}^{\dagger}\left(\mathbf{F},\underline{\mathbf{F}},G\right)\delta^2 S\left(\bar{\Psi},\hat{\Psi}\right)\mathbf{\Lambda}\left(\mathbf{F},\underline{\mathbf{F}},G\right) \end{aligned} \tag{52}$$

where $\delta^2 S\left(\bar{\Psi},\hat{\Psi}\right)$ is the second order derivative of the action functional for banks and investors fields:

$$\delta^2 S\left(\bar{\Psi},\hat{\Psi}\right)=\frac{\delta^2\left(\hat{S}\left(\hat{\Psi}\right)+\bar{S}\left(\bar{\Psi}\right)\right)}{\delta\hat{\Psi}\left(\hat{K},X\right)\delta\hat{\Psi}^{\dagger}\left(\hat{K},X\right)}+\frac{\delta^2\bar{S}\left(\bar{\Psi}\right)\left(\hat{S}\left(\hat{\Psi}\right)+\bar{S}\left(\bar{\Psi}\right)\right)}{\delta\bar{\Psi}\left(\bar{K},X\right)\delta\bar{\Psi}^{\dagger}\left(\bar{K},X\right)}$$

This action functional corresponds to a fiber deviation field theory:

$$\left[\begin{array}{c}\mathbf{\Lambda}\left(\mathbf{F},\underline{\mathbf{F}},G\right)\\ \delta^2 S\left(\bar{\Psi},\hat{\Psi}\right)\end{array}\right]_{[\mathbf{S},G]} \tag{53}$$

Gathering fiber equilibrium and deviation fields The field-theoretical framework that jointly considers the equilibrium fiber field theory (51) and the fiber deviation field $\mathbf{\Lambda}\left(\mathbf{F},\underline{\mathbf{F}},G\right)$ is given by the sum of the action functionals:

$$S^F=\hat{S}_{[\mathbf{S}]}\left(\hat{\Psi}\left(K,X\right)\right)+\bar{S}_{[\mathbf{S}]}\left(\hat{\Psi}\left(K,X\right),\bar{\Psi}\left(K,X\right)\right)+\mathbf{\Lambda}^{\dagger}\left(\mathbf{F},\underline{\mathbf{F}},G\right)\delta^2 S\left(\bar{\Psi},\hat{\Psi}\right)\mathbf{\Lambda}\left(\mathbf{F},\underline{\mathbf{F}},G\right) \tag{54}$$

corresponding to the field theory gathering (51) and (53):

$$\left[\begin{array}{c}\hat{\Psi},\bar{\Psi},\mathbf{\Lambda}\left(\mathbf{F},\underline{\mathbf{F}},G\right)\\ \hat{S}\left(\hat{\Psi}\right),\bar{S}\left(\hat{\Psi},\bar{\Psi}\right),S_F\left(\mathbf{\Lambda}\right)\end{array}\right]_{[\mathbf{S},G]} \tag{55}$$

9.3 Dynamics in Fibered Deviation States

The full dynamics of the system should incorporate both basis and fiber deviation states and the combination of these states leads to the notion of fibered deviation states.

9.3.1 Fibered Deviation States

By combining basis and fiber states, a complete description of fibered deviation states must therefore, as in the case of basis deviation fields, distinguish between the number of groups and the nature of their interactions.

[47] The expressions of the field action functionals $\hat{S}_{[\mathbf{S}]}\left(\hat{\Psi}\right)$ for the investors' field, and $\bar{S}_{[\mathbf{S}]}\left(\bar{\Psi},\hat{\Psi}\right)$ for banks' fields are given in Appendix 1.

Single group The deviation states for a single group should account for the level of disposable capital, sectoral background fields and return, and be written as:

$$\begin{bmatrix} |\mathbf{F},\underline{\mathbf{F}},G\rangle_{[\mathbf{S}]} \\ \updownarrow \\ |[\mathbf{S},\underline{\mathbf{S}},G]\rangle \end{bmatrix} \tag{56}$$

where the notation $\left[\updownarrow\right]$ indicates that any state of the system is composed of two interacting deviation states, basis and fiber.

This notation can be read at two levels.

First, the level of the basis state, which describes a basis deviation state for the stakes of group G:

$$|[\mathbf{S},\underline{\mathbf{S}},G]\rangle$$

Second, the level of the fiber deviation state, which describes one possible fiber deviation state, here capital, sectoral background fields and returns[48] for investors, banks and firms, for the given stakes:

$$|\mathbf{F},\underline{\mathbf{F}},G\rangle_{[\mathbf{S}]}$$

These possible fiber deviation states are described by the fields $\Psi(K,X)$, $\hat{\Psi}\left(\hat{K},X\right)$ and $\bar{\Psi}\left(\bar{K},X\right)$ minimizing the action functionals $\hat{S}\left(\hat{\Psi}\left(\hat{K},X\right)\right)$ and $\bar{S}\left(\hat{\Psi}\left(\hat{K},X\right),\bar{\Psi}\left(\bar{K},X\right)\right)$ with average fiber variables $\underline{\mathbf{F}}$ and by deviations $\mathbf{F}$ from these averages.

Multiple independent groups The description of the full dynamic collective state of multiple independent groups is given by the product of independent states:

$$\prod_G \begin{bmatrix} |\mathbf{F},\underline{\mathbf{F}},G\rangle_{[\mathbf{S}]} \\ \updownarrow \\ |[\mathbf{S},\underline{\mathbf{S}},G]\rangle \end{bmatrix} \tag{57}$$

Here again, this notation can be read at two levels.

First, at the level of the basis deviation state. It describes the product of all the basis deviation states:

$$\prod_G |[\mathbf{S},\underline{\mathbf{S}},G]\rangle$$

Second, at the level of the fiber deviation state. It describes the product of all the fiber deviation states:

$$\prod_G |\mathbf{F},\underline{\mathbf{F}},G\rangle_{[\mathbf{S}]}$$

for a given set of values of the basis variables:

$$(\mathbf{S},\underline{\mathbf{S}})$$

Multiple interacting groups For collective states composed of multiple interacting groups, we complete the above notation by accounting for the connections across a collection of groups:

$$\left[\prod_G [\mathbf{S},G], \prod_{G,G'} \mathbf{S}_{G,G'}\right]$$

[48] They are described in Gosselin and Lotz (2024).

where $[\mathbf{S}, G]$ denotes the stakes within groups, and $\mathbf{S}_{G,G'}$ denotes the stakes between groups G and G', or, in a more compact form:

$$[\mathbf{S}, \{G, G'\}]$$

where the collection $\{G, G'\}$ refers to the group interactions.

Without clusters When all groups interact with each others, a deviation state includes these interactions and involves all interacting basis and fiber deviation states as a composite deviation state. It will thus be written:

$$\begin{bmatrix} \prod_G |\mathbf{F}, \underline{\mathbf{F}}, G\rangle_{[\mathbf{S},\{G,G'\}]} \\ \updownarrow \\ \overbrace{\prod_G |[\mathbf{S}, \underline{\mathbf{S}}, G]\rangle}^{\mathbf{S}_{G,G'}} \end{bmatrix} \tag{58}$$

where the expression:

$$\prod_G |\mathbf{F}, \underline{\mathbf{F}}, G\rangle_{[\mathbf{S},\{G,G'\}]} \tag{59}$$

is the fiber deviation state above the cluster of basis deviation states:

$$\overbrace{\prod_G |[\mathbf{S}, \underline{\mathbf{S}}, G]\rangle}^{\mathbf{S}_{G,G'}}$$

that were described in (44).

Including clusters However, when groups are organized in independent clusters, the deviation states are products of independent clusters of interacting deviation states, and the notation becomes:

$$\prod_{Cl} \begin{bmatrix} \prod_{G\in Cl} |\mathbf{F}, \underline{\mathbf{F}}, G\rangle_{[\mathbf{S},\{G\in Cl, G'\in Cl\}]} \\ \updownarrow \\ \overbrace{\prod_{G\in Cl} |[\mathbf{S}, \underline{\mathbf{S}}, G]\rangle}^{\mathbf{S}_{G,G'}} \end{bmatrix} \tag{60}$$

where the product $\prod_{Cl}$ is over the clusters composing the collective state.

9.3.2 Fibered deviation fields

The fibered states (58) and (60) should be described by a field theory gathering both the basis deviation fields and the fiber equilibrium and deviation fields. We will start by describing the fibered deviation fields for one group, then for multiple independent groups, and multiple interacting groups.

Single group The complete description encompassing both basis and fiber deviation states is obtained by accounting for all possible levels of the basis variable and then combining the fiber and basis deviation fields.

The fibered deviation states for one group G defined in (56) should therefore emerge from a field-theoretical framework that jointly considers the fiber field theory (55) and the basis field theory (47) to which it is attached. We refer to this unified framework as the *fibered field theory*, which we denote as:

$$\left[\begin{array}{c}\left[\begin{array}{c}\hat{\Psi},\bar{\Psi},\mathbf{\Lambda}\left(\mathbf{F},\underline{\mathbf{F}},G\right)\\ \hat{S}\left(\hat{\Psi}\right),\bar{S}\left(\hat{\Psi},\bar{\Psi}\right),S_{F}\left(\mathbf{\Lambda}\right)\end{array}\right]_{[\mathbf{S},G]}\\ \updownarrow\\ \left[S\left(\Lambda\left(\mathbf{S},\underline{\mathbf{S}}\right),G\right)\right]\end{array}\right] \tag{61}$$

This expression describes a field theory involving both basis and fiber fields. In this expression, the symbol $\updownarrow$ indicates that the relationship between the two theories is not unidirectional, and that the fields $\hat{\Psi}\left(\hat{K},X\right)$, $\bar{\Psi}\left(\bar{K},X\right)$ and $\mathbf{\Lambda}\left(\mathbf{F},\underline{\mathbf{F}},G\right)$ are conditioned by the basis states and the values of sectoral stakes.

Conversely, the fiber states also influence the basis states. Indeed, some fiber transitions can trigger basis transitions. In the present model, the action functional of this fibered field theory should therefore be the sum of the action functional for basis deviation fields (46) and the action functional for fiber equilibrium and deviation fields (54):

$$S\left(\Lambda\left(\mathbf{S},\underline{\mathbf{S}},G\right)\right)+\hat{S}_{[\mathbf{S}]}\left(\hat{\Psi}\left(K,X\right)\right)+\bar{S}_{[\mathbf{S}]}\left(\hat{\Psi}\left(K,X\right),\bar{\Psi}\left(K,X\right)\right)+\mathbf{\Lambda}^{\dagger}\left(\mathbf{F},\underline{\mathbf{F}},G\right)\delta^{2}S\left(\bar{\Psi},\hat{\Psi}\right)\mathbf{\Lambda}\left(\mathbf{F},\underline{\mathbf{F}},G\right) \tag{62}$$

In expression (61), the fiber fields form a superstructure determined by the landscape of connections, i.e. the stakes. Consequently, the full field theory for collective states should account both for modifications of the basis—as outlined in the previous section describing transitions between groups—and for changes in the superstructure, which may switch from one state to another and thereby modify the underlying basis.

Thus, collective states are, in fact, states of stakes with an associated fiber, each fiber being itself a share-dependent field theory. This fibered field theory, in turn, describes the possible states of an economy, with agents interacting according to their dynamics, potential defaults, and transitions arising from the diffusion of returns and disposable capital.

Multiple independent groups For multiple independent groups, the previous argument generalizes directly, and the states (57) are described an action action functional which is the sum of the action functionals of each group:

$$\begin{aligned}&\sum_{G}\left(S\left(\Lambda\left(\mathbf{S},\underline{\mathbf{S}},G\right)\right)+\hat{S}_{[\mathbf{S},G]}\left(\hat{\Psi}\left(K,X\right)\right)+\bar{S}_{[\mathbf{S},G]}\left(\hat{\Psi}\left(K,X\right),\bar{\Psi}\left(K,X\right)\right)\right)\\&+\sum_{G}S_{F}\left(\left\{\mathbf{\Lambda}\left(\mathbf{F},\underline{\mathbf{F}},G\right)\right\}_{G}\right)\end{aligned} \tag{63}$$

This first part of the action functional corresponds to the sum of individual groups actions (62), while the second part is obtained a a sum of terms (52):

$$\begin{aligned}&S_{F}\left(\left\{\mathbf{\Lambda}\left(\mathbf{F},\underline{\mathbf{F}},G\right)\right\}_{G}\right)\\=&\sum_{G}\mathbf{\Lambda}^{\dagger}\left(\mathbf{F},\underline{\mathbf{F}},G\right)\delta^{2}S\left(\bar{\Psi},\hat{\Psi}\right)\mathbf{\Lambda}\left(\mathbf{F},\underline{\mathbf{F}},G\right)\end{aligned} \tag{64}$$

where $\left\{\mathbf{\Lambda}\left(\mathbf{F},\underline{\mathbf{F}},G\right)\right\}_{G}$ is the collection of deviation fiber fields for the whole set of groups.

The action functional (63) corresponds to a fibered field theory which is the product of several independent fibered field theories:

$$\prod_{G}\left[\begin{array}{c}\left[\begin{array}{c}\hat{\Psi},\bar{\Psi},\mathbf{\Lambda}\left(\mathbf{F},\underline{\mathbf{F}},G\right)\\ \hat{S}\left(\hat{\Psi}\right),\bar{S}\left(\hat{\Psi},\bar{\Psi}\right),S_{F}\left(\mathbf{\Lambda}\right)\end{array}\right]_{[\mathbf{S},G]}\\ \updownarrow\\ \left[S\left(\Lambda\left(\mathbf{S},\underline{\mathbf{S}}\right),G\right)\right]\end{array}\right] \tag{65}$$

Multiple interacting groups Recall that the full deviation states for multiple interacting groups are given by a combination of the form (58). As before, the fiber states in (58) for investors and banks are defined over the distribution $[\mathbf{S}, \{G, G'\}]$:

$$|\mathbf{F}, \underline{\mathbf{F}}\rangle_{[\mathbf{S},\{G,G'\}]}$$

and should arise from a field theory similar to (65). However, as with the states, since the stakes $[\mathbf{S}, \{G, G'\}]$ connect the various groups, we cannot consider the full theory describing interacting groups as a product of independent field theories. For each basis state:

$$\prod_{Cl} \overbrace{\prod_{G \in Cl} |[\mathbf{S}, \underline{\mathbf{S}}, G]\rangle}^{\mathbf{S}_{G,G'}}$$

we should consider the associated field theory as being described by a fully interacting set of fiber fields:

$$\prod_{Cl} \left[\begin{array}{c} \prod_{G \in Cl} \left[\begin{array}{c} \hat{\Psi}, \bar{\Psi}, \mathbf{\Lambda}(\mathbf{F}, \underline{\mathbf{F}}, G) \\ \hat{S}\left(\hat{\Psi}\right), \bar{S}\left(\hat{\Psi}, \bar{\Psi}\right), S_F(\mathbf{\Lambda}) \end{array} \right]_{[\mathbf{S},\{G,G'\}]} \\ \updownarrow \\ \overbrace{\prod_{G \in Cl} |[\mathbf{S}, \underline{\mathbf{S}}, G]\rangle}^{\mathbf{S}_{G,G'}} \end{array} \right] \tag{66}$$

Such states should arise from a fibered field theory encompassing both basis and fiber fields:

$$\prod_{Cl} \left[\begin{array}{c} \left[\prod_{G \in Cl} \left[\begin{array}{c} \hat{\Psi}, \bar{\Psi}, \mathbf{\Lambda}(\mathbf{F}, \underline{\mathbf{F}}, G) \\ S\left(\left\{\hat{\Psi}, \bar{\Psi}, \mathbf{\Lambda}\right\}\right) \end{array} \right]_{[\mathbf{S},\{G,G'\}]} \right] \\ \updownarrow \\ \left[\prod_{G \in Cl} \left[S\left(\left\{\Lambda\left(\mathbf{S}, \underline{\mathbf{S}}, G\right)\right\}\right)\right]\right] \end{array} \right] \tag{67}$$

where both the basis theory:

$$\left[\prod_{G \in Cl} \left[S\left(\left\{\Lambda\left(\mathbf{S}, \underline{\mathbf{S}}, G\right)\right\}\right)\right]\right]$$

and fiber theory:

$$\left[\prod_{G \in Cl} \left[\begin{array}{c} \hat{\Psi}, \bar{\Psi}, \mathbf{\Lambda}(\mathbf{F}, \underline{\mathbf{F}}, G) \\ S\left(\left\{\hat{\Psi}, \bar{\Psi}, \mathbf{\Lambda}\right\}\right) \end{array} \right]_{[\mathbf{S},\{G,G'\}]} \right]$$

Condition each other, which justifies the symbol $\updownarrow$ in (67) to indicate that the relationship between the two theories is not unidirectional and that the fiber also influences the basis states. The use of curly brackets {} in the notations:

$$S\left(\left\{\Lambda\left(\mathbf{S}, \underline{\mathbf{S}}, G\right)\right\}\right)$$

and:

$$S\left(\left\{\hat{\Psi}, \bar{\Psi}, \mathbf{\Lambda}\right\}\right)$$

emphasizes the presence of intra-cluster interactions: the basis and fiber action functionals must incorporate not only the action functionals $S\left(\Lambda\left(\mathbf{S}, \underline{\mathbf{S}}, G\right)\right)$ and $S\left(\hat{\Psi}, \bar{\Psi}, \mathbf{\Lambda}\right)$ of the individual groups, but also the interactions between groups, which involve the entire set of basis deviation and fiber fields.

Such a field theory should be represented by an action functional of the form:

$$S^{B,F} = \sum_{G} S\left(\Lambda\left(\mathbf{S}, \underline{\mathbf{S}}\right), G\right) + \sum_{G,G'} S_{G,G'}\left(\left(\Lambda\left(\mathbf{S}^{(G/G')}, \underline{\mathbf{S}}^{(G/G')}\right)\right)\right) \tag{68}$$
$$+ \sum_{G} \hat{S}_{[\mathbf{S},\{G,G'\}]}\left(\hat{\Psi}\right) + \sum_{G} \bar{S}_{[\mathbf{S},\{G,G'\}]}\left(\hat{\Psi}, \bar{\Psi}\right)$$
$$+ \sum_{G,G'} \delta S_{G,G'}\left(\left(\hat{\Psi}^{(G)}, \bar{\Psi}^{(G)}\right), \left(\hat{\Psi}^{(G')}, \bar{\Psi}^{(G')}\right)\right)$$
$$+ \sum_{G} \mathbf{\Lambda}^{\dagger}\left(\mathbf{F}, \underline{\mathbf{F}}, G\right) \delta^2 S\left(\bar{\Psi}, \hat{\Psi}\right) \mathbf{\Lambda}\left(\mathbf{F}, \underline{\mathbf{F}}, G\right)$$
$$+ \sum_{G,G'} \delta^2 S_{G,G'}\left(\bar{\Psi}, \hat{\Psi}, \mathbf{\Lambda}\left(G\right), \mathbf{\Lambda}'\left(G\right)\right)$$

where the fourth term:

$$\sum_{G,G'} \delta S_{G,G'}\left(\left(\hat{\Psi}^{(G)}, \bar{\Psi}^{(G)}\right), \left(\hat{\Psi}^{(G')}, \bar{\Psi}^{(G')}\right)\right)$$

describes the interactions between the fibers of the various groups and the last terms:

$$\sum_{G,G'} \delta^2 S_{G,G'}\left(\bar{\Psi}, \hat{\Psi}, \mathbf{\Lambda}\left(G\right), \mathbf{\Lambda}'\left(G\right)\right)$$

the interactions between deviations around the equilibrium values of fiber variables. The possible forms of these interaction terms will be considered in subsequent work on the subject.

In formula (68), the expression:

$$\hat{S}_{[\mathbf{S},\{G,G'\}]}\left(\hat{\Psi}\left(\hat{K}, X\right)\right)$$

describes the action functional for the investors and:

$$\bar{S}_{[\mathbf{S},\{G,G'\}]}\left(\hat{\Psi}\left(\hat{K}, X\right), \bar{\Psi}\left(\bar{K}, X\right)\right)$$

is the action functional for banks fields. They are given by formula (92) and (93) in Appendix 1. The stakes involved in these formula are set to the values $[\mathbf{S}, \{G, G'\}]$ and the second term in (68) describes the interactions between basis states.

To conclude, note that:

1) The hypothetical field theory operates on a much larger space than the various fields:

$$\prod_{G} \left[S\left(\Lambda\left(\mathbf{S}, \underline{\mathbf{S}}\right), G\right)\right]$$

since such field theories are themselves variables and should be considered as the arguments of a field of fields.

2) The total probability for the collective state (58) should be:

$$\exp\left(-S_{full}\left(\Lambda\left(\mathbf{S}, \underline{\mathbf{S}}\right), \hat{\Psi}, \bar{\Psi}, \mathbf{\Lambda}\left(\mathbf{F}, \underline{\mathbf{F}}, G\right), G\right)\right) \tag{69}$$

where S_{full} is defined by (68) and includes the fiber variables.

10 Transitions Between Collective States

Since collective states are composed of interacting groups of agents in specific phases, they can be considered dynamic in the long run. Even in the short run, they may experience fluctuations that do not alter the intrinsic characteristics of the groups but may change their phases at any time.

In this context, instability may arise from fluctuations or exogenous modifications in the system. It may also occur when the strength of agents' interactions within a group varies-for instance, when some agents strengthen their connections at the expense of others.

Moreover, the number of groups within the system may itself vary. Any initial structure of the system may either expand or contract. For instance, emerging groups may merge with existing ones, thereby modifying the stability of the system. These various patterns were not captured in our initial description of fluctuations around static collective states and the following paragraphs will fill this gap by exploring the various types of transitions for basis and fibers that could be expected. This typology will provide a framework to develop later a field theory for sub-collective states, that is, for groups and the phases they occupy accounting for the possible pattrns f transitions.

10.1 Transitions for Basis States

In collective states, transitions stem from interactions between agents-here the stakes[49]. Within a given group, circular over-investments across several sectors may induce instability and trigger the entire group toward another, possibly default, state. This occurs when investment returns in one sector significantly exceed the firms' returns, reflecting the fact that investors favour cross-sector investment returns at the expense of their own sector's capital expenditures. In this case, any modification in cross- sector investment returns may lead to a sequence of defaults across the entire group[50].

More generally, interactions betweeen groups may trigger transition mechanisms involving several groups. These transitions may induce some default states, but also the modifications in the composition of the groups, including merging or disscociatin of groups.

We detail different types of transitions in the case of a single group, multiple independent or interacting grps..

10.1.1 Transitions for a single group

Due to internal dynamics, or external perturbations, a sub-collective state may experience a transition. In general, this transition does not affect the group, since in this case it is unique, but modifies the phase it is in. Thus, the transition only affects the phase of the sub-collective state. A transition will only modify the associated group in the default mechanism, in which the number of sectors diminishes.

Without modification of the group In general, for a given level of uncertainty and a given stability range, a deviation state undergoes a transition of its phase $\underline{\mathbf{S}}$, without any change in the structure of the group G. Considering a deviation state:

$$|[\mathbf{S}, \underline{\mathbf{S}}, G]\rangle$$

this deviation state becomes unstable when fluctuations $\mathbf{S}$ move outside the stability range around $\underline{\mathbf{S}}$. Fluctuations may be initiated by some exogenous modifications and induce a transition between

[49]Technically, instability and transitions arise from modification of eigenvalues due to changes in connections.

[50]See Gosselin and Lotz (2025a,b).

deviation states, so that:

$$|[\mathbf{S},\underline{\mathbf{S}},G]\rangle \rightarrow \left|\left[\mathbf{S}',\underline{\mathbf{S}}',G\right]\right\rangle$$

where the phase $\underline{\mathbf{S}}$ of the group G shifts to a new phase $\underline{\mathbf{S}}'$.

With modification of the group For a single independent group, the only modification of the group arising from internal fluctuations is one among several possible default states. Such transition toward defaults can be writen as:

$$|[\mathbf{S},\underline{\mathbf{S}},G]\rangle \rightarrow \left|\left[\mathbf{S}',\underline{\mathbf{S}}^{DF},G/DF\right]\right\rangle$$

where G/DF is the set of remaining sectors after default.

10.1.2 Transitions of multiple independent groups

Without modification of groups When several independent groups compose a collective state, internal transitions occuring in multiple groups may be considered, and the transition can be written as:

$$\prod_G |[\mathbf{S},\underline{\mathbf{S}},G]\rangle \rightarrow \prod_G \left|\left[\mathbf{S}',\underline{\mathbf{S}}',G\right]\right\rangle$$

Each of these transitions arise independently from the other.

With modification of groups The general form of such transition is:

$$\prod_G |[\mathbf{S},\underline{\mathbf{S}},G]\rangle \rightarrow \prod_{G'} \left|\left[\mathbf{S}',\underline{\mathbf{S}}',G'\right]\right\rangle$$

where a product of states corresponding to a given collection of groups is transformed into a product associated with another collection of groups. We will specifically presetn here the case of mergers and dissociations.

Mergers Groups may merge, which corresponds to the following transition in terms of states:

$$\prod_{G_1} |[\mathbf{S}_1,\underline{\mathbf{S}}_1,G_1]\rangle \prod_{G_2} |[\mathbf{S}_2,\underline{\mathbf{S}}_2,G_2]\rangle \rightarrow \prod_G |[\mathbf{S},\underline{\mathbf{S}},G]\rangle$$

where G denotes the aggregation of two or more groups. This binding between groups can be described more precisely in terms of stakes. Considering two merging groups:

$$|[\mathbf{S}_1,\underline{\mathbf{S}}_1,G_1]\rangle\,|[\mathbf{S}_2,\underline{\mathbf{S}}_2,G_2]\rangle \rightarrow |\mathbf{S},\underline{\mathbf{S}},G_1\cup G_2\rangle$$

the deviation stakes and the phase:

$$(\mathbf{S},\underline{\mathbf{S}})$$

are modifications in the deviations and phases for the two groups:

$$(\mathbf{S}_1,\underline{\mathbf{S}}_1)\,,(\mathbf{S}_2,\underline{\mathbf{S}}_2)$$

together with some inter-groups connections $\underline{S}_{12}$. Such transitions may be induced either by direct interactions between the two groups or by the temporary binding of a third group with G_1 and G_2.

$$\begin{aligned} & |[\mathbf{S}_1,\underline{\mathbf{S}}_1,G_1]\rangle\,|[\mathbf{S}_3,\underline{\mathbf{S}}_3,G_3]\rangle \times |[\mathbf{S}_2,\underline{\mathbf{S}}_2,G_2]\rangle\,|[\mathbf{S}_3,\underline{\mathbf{S}}_3,G_3]\rangle \\ \rightarrow\ & |[\mathbf{S},\underline{\mathbf{S}},G_1\cup G_2\cup G_3]\rangle \rightarrow (|[\mathbf{S}_1,\underline{\mathbf{S}}_1,G_1]\rangle\,|[\mathbf{S}_2,\underline{\mathbf{S}}_2,G_2]\rangle)\,|[\mathbf{S}_3,\underline{\mathbf{S}}_3,G_3]\rangle \end{aligned}$$

Dissociation Several groups may also split, which corresponds to the following transitions:

$$\prod_{G}|[\mathbf{S},\underline{\mathbf{S}},G]\rangle\rightarrow\prod_{G_1}|[\mathbf{S}_1,\underline{\mathbf{S}}_1,G_1]\rangle\prod_{G_2}|[\mathbf{S}_2,\underline{\mathbf{S}}_2,G_2]\rangle$$

This describes the reverse mechanism of merging states. For two groups:

$$|\mathbf{S},\underline{\mathbf{S}},G_1\cup G_2\rangle\rightarrow|[\mathbf{S}_1,\underline{\mathbf{S}}_1,G_1]\rangle\,|[\mathbf{S}_2,\underline{\mathbf{S}}_2,G_2]\rangle$$

this corresponds to inter-groups connections $\underline{S}_{12}$ that gradually fade away.

Defaults When transitions toward default state arise for some of these independent groups, this can be writen as:

$$\prod_{G}|[\mathbf{S},\underline{\mathbf{S}},G]\rangle\rightarrow\prod_{G}\left|\left[\mathbf{S}',\underline{\mathbf{S}}',G\right]\right\rangle\prod_{G}\left|\left[\mathbf{S}',\underline{\mathbf{S}}^{DF},G/DF\right]\right\rangle$$

where the first product involves the non-defaulting groups, and the second involves the defaulting groups.

Uncertainty-induced In the formation of groups, the uncertainties ζ^2 and $\hat{h}_1^{(0)}(X',X_{m-1})$ about investments play an important role, as they determine the stakes through the coefficients $\hat{w}(X',X)$. When ζ^2 or $\hat{w}_1^{(0)}(X',X_{m-1})$ decrease within a given group, the connections satisfying the saddle-point equations are modified, which may trigger shifts in the system corresponding to a reorganization of groups. In terms of transitions, this can be written as:

$$\prod_{G_1}|[\mathbf{S}_1,\underline{\mathbf{S}}_1,G_1]\rangle\rightarrow\prod_{G_2}|[\mathbf{S}_2,\underline{\mathbf{S}}_2,G_2]\rangle$$

where the indices 1 and 2 refer to different organization in groups and stakes. As a result, the forms of several groups are modified.

10.1.3 Transitions of multiple interacting groups

When groups interact in clusters, the fluctuations may propagate from one group to another, leading to several simultaneous transitions. The mechanisms described above still apply, but must be adapted to account for interactions. In the following, we will focus on the general case involving multiple clusters of states, in which the case with a single cluster is a particular case.

Without modification of groups The principle is the same as before. The decomposition in clusters remains globally unchanged, but inside each cluster, groups experience a change in their level of stakes, $\mathbf{S}$, and in their phase $\underline{\mathbf{S}}$, so that the corresponding transition writes:

$$\prod_{Cl}\left(\overbrace{\prod_{G\in Cl}|[\mathbf{S},\underline{\mathbf{S}},G]\rangle}^{\mathbf{S}_{G\in Cl,G'\in Cl}}\right)\rightarrow\prod_{Cl}\left(\overbrace{\prod_{G\in Cl}\left|\left[\mathbf{S}',\underline{\mathbf{S}}',G\right]\right\rangle}^{\mathbf{S}'_{G\in Cl,G'\in Cl}}\right)\tag{70}$$

However, in the present setting, transitions within a cluster are not independent and may depend on interactions between groups. Fluctuations of a given group, although they do not modify the phase of that group, may induce transitions in neighboring groups. These inter-group transition mechanisms will be studied in greater detail in a future work.

With modification of groups Among interacting groups, the transitions associated with modifications in groups are similar to those observed in the case of multiple independent groups. However, we can distinguish between modifications that occur within a cluster and those that occur between clusters.

Intra-clusters modifications Here again, the groups of a cluster may merge or dissociate, due to modification of uncertainty or interactions. The two mechanisms are reciprocal, so that:

$$\left(\overbrace{\prod_{G_1\in Cl}\left|\left[\mathbf{S}_1,\underline{\mathbf{S}}_1,G_1\right]\right\rangle\prod_{G_2\in Cl}\left|\left[\mathbf{S}_2,\underline{\mathbf{S}}_2,G_2\right]\right\rangle}^{\mathbf{S}_{G\in Cl,G'\in Cl}}\right)\rightleftharpoons\left(\overbrace{\prod_{G\in Cl}\left|\left[\mathbf{S},\underline{\mathbf{S}},G\right]\right\rangle}^{\mathbf{S}_{G\in Cl,G'\in Cl}}\right)$$

where the double arrow $\rightleftharpoons$ accounts for the possibility of transition in both directions.

Inter-clusters modifications Uncertainty can induce reorganization of groups, such as clusters merging or dissociating. This type of transitions are written:

$$\left(\overbrace{\prod_{G_1\in Cl}\left|\left[\mathbf{S}_1,\underline{\mathbf{S}}_1,G_1\right]\right\rangle}^{\mathbf{S}_{G\in Cl,G'\in Cl}}\overbrace{\prod_{G_2\in Cl}\left|\left[\mathbf{S}_2,\underline{\mathbf{S}}_2,G_2\right]\right\rangle}^{\mathbf{S}_{G\in Cl,G'\in Cl}}\right)\rightleftharpoons\left(\overbrace{\prod_{G\in Cl}\left|\left[\mathbf{S},\underline{\mathbf{S}},G\right]\right\rangle}^{\mathbf{S}_{G\in Cl,G'\in Cl}}\right)$$

Defaults Among the transitions induced by the interactions, some groups of a cluster may be driven to a default state, and the general form of such type of transition is:

$$\prod_{Cl}\left(\overbrace{\prod_{G\in Cl}\left|\left[\mathbf{S},\underline{\mathbf{S}},G\right]\right\rangle}^{\mathbf{S}_{G\in Cl,G'\in Cl}}\right)\to\prod_{Cl}\left(\overbrace{\prod_{G\in Cl}\left|\left[\mathbf{S}',\underline{\mathbf{S}}',G\right]\right\rangle\prod_{G\in Cl}\left|\left[\mathbf{S}',\underline{\mathbf{S}}^{DF},G/DF\right]\right\rangle}^{\mathbf{S}_{G\in Cl,G'\in Cl}}\right) \tag{71}$$

Uncertainty-induced We have seen that uncertainty may modify the composition of groups and thus, in turn, the form of the clusters: groups may reorganize and change the decomposition of the collective state into clusters. The mechanisms involved are similar to the case of independent groups, and the corresponding clusters' transition writes:

$$\prod_{Cl_1}\left(\overbrace{\prod_{G_1\in Cl_1}\left|\left[\mathbf{S}_1,\underline{\mathbf{S}}_1,G_1\right]\right\rangle}^{\mathbf{S}_{G_1\in Cl,_1G_1'\in Cl_1}}\right)\to\prod_{Cl_2}\left(\overbrace{\prod_{G_2\in Cl_2}\left|\left[\mathbf{S}_2,\underline{\mathbf{S}}_2,G_2\right]\right\rangle}^{\mathbf{S}_{G_2\in Cl,_2G_2'\in Cl_2}}\right) \tag{72}$$

10.1.4 Probabilities of transitions for basis states

Principle The transitions studied above are quantified by the *probabilities of transition*, which compute the probability that a transition occurs from an initial state:

$$\prod_{G_1}\left|\left[\mathbf{S}_1,\underline{\mathbf{S}}_1,G_1\right]\right\rangle$$

toward a final state:

$$\prod_{G_2} \left|[\mathbf{S}_2, \underline{\mathbf{S}}_2, G_2]\right\rangle$$

Formally, such a probability is written as:

$$\prod_{G_1} \left\langle[\mathbf{S}_1, \underline{\mathbf{S}}_1, G_1]\right| \exp\left(-S\right) \prod_{G_2} \left|[\mathbf{S}_2, \underline{\mathbf{S}}_2, G_2]\right\rangle \tag{73}$$

where the action functional S is given by (49):

$$S = \sum_G S\left(\Lambda\left(\mathbf{S}, \underline{\mathbf{S}}, G\right)\right) + \sum_{G,G'} \delta S_{G,G'}\left(\Lambda\left(\mathbf{S}, \underline{\mathbf{S}}, G\right), \Lambda\left(\mathbf{S}, \underline{\mathbf{S}}, G\right)\right) \tag{74}$$

For example, in the case of merging groups, the probability of transition takes the form:

$$\prod_{G} \left\langle[\mathbf{S}, \underline{\mathbf{S}}, G]\right| \exp\left(-S\right) \prod_{G_1} \left|[\mathbf{S}_1, \underline{\mathbf{S}}_1, G_1]\right\rangle \prod_{G_2} \left|[\mathbf{S}_2, \underline{\mathbf{S}}_2, G_2]\right\rangle$$

Conversely, for splitting groups, the probability of transition becomes:

$$\prod_{G_1} \left\langle[\mathbf{S}_1, \underline{\mathbf{S}}_1, G_1]\right| \prod_{G_2} \left\langle[\mathbf{S}_2, \underline{\mathbf{S}}_2, G_2]\right| \exp\left(-S\right) \prod_{G} \left|[\mathbf{S}, \underline{\mathbf{S}}, G]\right\rangle$$

Derivation The derivation of such probability of transition was presented in Gosselin and Lotz (2024). Formally, the probability (73) is computed as:

$$\begin{aligned} &\prod_{G_1} \left\langle[\mathbf{S}, \underline{\mathbf{S}}, G_1]\right| \exp\left(-S\right) \prod_{G_2} \left|[\mathbf{S}_2, \underline{\mathbf{S}}_2, G_2]\right\rangle \\ = &\int_{\substack{\left(\Lambda_{-\infty}, \Lambda^{\dagger}_{-\infty}\right) = \left|[\mathbf{S}_1, \underline{\mathbf{S}}_1, G_1]\right\rangle \\ \left(\Lambda_{-\infty}, \Lambda^{\dagger}_{-\infty}\right) = \left|[\mathbf{S}_2, \underline{\mathbf{S}}_2, G_2]\right\rangle}} \exp\left(-\int S_t dt\right) D\Lambda_t D\Lambda_t^{\dagger} \end{aligned} \tag{75}$$

Here Λ_t and $\Lambda_t^{\dagger}$ are extensions of the fields Λ and $\Lambda^{\dagger}$ including an explicit time variable. The action functional S_t is the functional S evaluated at Λ_t and $\Lambda_t^{\dagger}$:

$$S_t = \sum_G S\left(\Lambda_t\left(\mathbf{S}, \underline{\mathbf{S}}, G\right)\right) + \sum_{G,G'} \delta S_{G,G'}\left(\Lambda_t\left(\mathbf{S}, \underline{\mathbf{S}}, G\right), \Lambda_t\left(\mathbf{S}, \underline{\mathbf{S}}, G\right)\right)$$

The measure $D\Lambda_t D\Lambda_t^{\dagger}$ represents the functional integration over all realizations of the fields Λ and $\Lambda^{\dagger}$. The integral is performed under the boundary conditions:

$$\begin{aligned} \left(\Lambda_{-\infty}, \Lambda^{\dagger}_{-\infty}\right) &= \left|[\mathbf{S}_1, \underline{\mathbf{S}}_1, G_1]\right\rangle \\ \left(\Lambda_{-\infty}, \Lambda^{\dagger}_{-\infty}\right) &= \left|[\mathbf{S}_2, \underline{\mathbf{S}}_2, G_2]\right\rangle \end{aligned}$$

that is, the fields realizations are constrained by background fields corresponding to the states $\left|[\mathbf{S}_1, \underline{\mathbf{S}}_1, G_1]\right\rangle$ and $\left|[\mathbf{S}_2, \underline{\mathbf{S}}_2, G_2]\right\rangle$ as boundary conditions. The techniques required to compute these probabilities will be developed in the fourth part of this series.

10.2 Transitions for Fiber States

In a dynamic context, the fiber variables, eventhough depending they depend on basis variables, are considered as independent variables. As a consequence, fiber states $|\mathbf{F},\underline{\mathbf{F}},G\rangle$ can themselves undergo two types of transitions. First, they can experience specific-fiber transitions, that describe internal modifications of the fiber, for a given basis state. Second, fiber states can experience fiber transitions induced by basis transitions.

Since each fiber deviation state is attached to a basis state, and as such depends on a given group, we will describe both type of fiber transitions for a given basis deviation state - and thus a given group G . Therefore, we restrict our attention to the case of single-group transition.

10.2.1 Types of transitions

We first describe the fiber specific transitions, that are independent of the basis, then the transitions induced by the basis.

Fiber-specifc transition Transitions in fiber states may be spontaneous or externally driven. Return equations generally admit multiple solutions for a given distribution of stakes for which transitions of the form:

$$|\mathbf{F},\underline{\mathbf{F}},G\rangle \rightarrow |\mathbf{F},\underline{\mathbf{F}},G\rangle' \tag{76}$$

may therefore occur without basis transitions.

Such transitions may also involve defaults for some groups. In this case, the transition:

$$|\mathbf{F},\underline{\mathbf{F}},G\rangle \rightarrow |\mathbf{F},\underline{\mathbf{F}},G\rangle^{DF} \tag{77}$$

should induce a corresponding basis transition:

$$|[\mathbf{S},\underline{\mathbf{S}},G]\rangle \rightarrow \left|\left[\mathbf{S}',\underline{\mathbf{S}}^{DF},G/DF\right]\right\rangle$$

Remark that transitions (76) may in turn induce transitions in the basis states:

$$|[\mathbf{S},\underline{\mathbf{S}},G]\rangle \rightarrow \left|\left[\mathbf{S}',\underline{\mathbf{S}}',G\right]\right\rangle$$

Basis-induced transition Reciprocally, fiber states may undergo transitions induced by changes in stakes:

$$|[\mathbf{S},\underline{\mathbf{S}},G]\rangle \rightarrow |[\mathbf{S}',\underline{\mathbf{S}},G]\rangle$$

which describe fluctuations, or transitions in stakes:

$$|[\mathbf{S},\underline{\mathbf{S}},G]\rangle \rightarrow \left|\left[\mathbf{S}',\underline{\mathbf{S}}',G\right]\right\rangle$$

Both modifications may induce fiber transitions:

$$|\mathbf{F},\underline{\mathbf{F}},G\rangle \rightarrow \left|\mathbf{F}',\underline{\mathbf{F}}',G\right\rangle \tag{78}$$

10.2.2 Probabilities of transition for fiber states

Principle As for the basis states, we can associate a probability of transition to each transition of a fiber state. For two fiber states:

$$\left|\mathbf{F},\underline{\mathbf{F}},G\right\rangle$$

and:

$$\left|\mathbf{F}',\underline{\mathbf{F}}',G\right\rangle$$

it is computed by evaluating:

$$\left\langle \mathbf{F}',\underline{\mathbf{F}}',G\right| \exp\left(-S^{F}\right)\left|\mathbf{F},\underline{\mathbf{F}},G\right\rangle \tag{79}$$

where the action functional S^F is given by (54).

Derivation The derivation of the probability of transition between fiber states (79) is similar to that of the probability of transition between basis states (75). Formally, it is computed as:

$$\begin{aligned}
&\left\langle \mathbf{F}',\underline{\mathbf{F}}',G\right| \exp\left(-S^{F}\right)\left|\mathbf{F},\underline{\mathbf{F}},G\right\rangle \\
=&\int_{\substack{\left(\Lambda_{-\infty},\Lambda_{-\infty}^{\dagger}\right)=\left|\mathbf{F},\underline{\mathbf{F}},G\right\rangle \\ \left(\Lambda_{-\infty},\Lambda_{-\infty}^{\dagger}\right)=\left|\mathbf{F}',\underline{\mathbf{F}}',G\right\rangle}} \exp\left(-\int S_{t}dt\right) D\mathbf{\Lambda}_{t}D\mathbf{\Lambda}_{t}^{\dagger}D\hat{\Psi}_{t}D\hat{\Psi}_{t}^{\dagger}D\bar{\Psi}_{t}D\bar{\Psi}_{t}^{\dagger}
\end{aligned} \tag{80}$$

Here $\mathbf{\Lambda}_t$, $\mathbf{\Lambda}_t^{\dagger}$, $\hat{\Psi}_t$, $\hat{\Psi}_t^{\dagger}$ and $\bar{\Psi}_t$, $\bar{\Psi}_t^{\dagger}$ are extensions of the fields $\mathbf{\Lambda}$, $\mathbf{\Lambda}^{\dagger}$, $\hat{\Psi}$, $\hat{\Psi}^{\dagger}$ and $\bar{\Psi}$, $\bar{\Psi}^{\dagger}$ including an explicit time variable. The action functional S_t^F is the functional S^F evaluated at $\left(\mathbf{\Lambda}_t,\mathbf{\Lambda}_t^{\dagger},\hat{\Psi}_t,\hat{\Psi}_t^{\dagger},\bar{\Psi}_t,\bar{\Psi}_t^{\dagger}\right)$:

$$S_t^F=\hat{S}_{[\mathbf{S}]}\left(\hat{\Psi}_t\left(K,X\right)\right)+\bar{S}_{[\mathbf{S}]}\left(\hat{\Psi}_t\left(K,X\right),\bar{\Psi}_t\left(K,X\right)\right)+\mathbf{\Lambda}_t^{\dagger}\left(\mathbf{F},\underline{\mathbf{F}},G\right)\delta^2 S\left(\bar{\Psi},\hat{\Psi}\right)\mathbf{\Lambda}_t\left(\mathbf{F},\underline{\mathbf{F}},G\right) \tag{81}$$

$$S_t=\sum_G S\left(\Lambda_t\left(\mathbf{S},\underline{\mathbf{S}},G\right)\right)+\sum_{G,G'}\delta S_{G,G'}\left(\Lambda_t\left(\mathbf{S},\underline{\mathbf{S}},G\right),\Lambda_t\left(\mathbf{S},\underline{\mathbf{S}},G\right)\right)$$

The measure:

$$D\mathbf{\Lambda}_{t}D\mathbf{\Lambda}_{t}^{\dagger}D\hat{\Psi}_{t}D\hat{\Psi}_{t}^{\dagger}D\bar{\Psi}_{t}D\bar{\Psi}_{t}^{\dagger}$$

represents the functional integration over all realizations of the fields:

$$\left[\mathbf{\Lambda}_t,\mathbf{\Lambda}_t^{\dagger}\right]^F\equiv\left(\mathbf{\Lambda}_t,\mathbf{\Lambda}_t^{\dagger},\hat{\Psi}_t,\hat{\Psi}_t^{\dagger},\bar{\Psi}_t,\bar{\Psi}_t^{\dagger}\right)$$

. The integral is performed under the boundary conditions:

$$\begin{aligned}
\left[\mathbf{\Lambda}_{-\infty},\mathbf{\Lambda}_{-\infty}^{\dagger}\right]^F &= \left|\mathbf{F},\underline{\mathbf{F}},G\right\rangle \\
\left[\mathbf{\Lambda}_{\infty},\mathbf{\Lambda}_{\infty}^{\dagger}\right]^F &= \left|\mathbf{F}',\underline{\mathbf{F}}',G\right\rangle
\end{aligned}$$

10.3 Transitions for Fibered states

Since fibered states describe the system in its globality, they encompass the transitions both of the basis and the fiber states. However, as basis transitions may induce fiber transitions, we can also consider that fiber transitions induce basis transitions. These reciprocal interactions imply possible sequences of retroaction and transitions between fibered states.

10.3.1 Transitions for a single group

Here, the two possible cases of basis transitions, without or with modification of group can be considered. However we have also to account that these transitions will induce fiber transitions. Moreover, a fiber transition may itself impact the basis and induce a transition for a full fibered state.

Without modification of the group In this case, the basis transition:

$$|[\mathbf{S}, \underline{\mathbf{S}}, G]\rangle \rightarrow |[\mathbf{S}', \underline{\mathbf{S}}', G]\rangle$$

induces a fiber transition (78), so that at the level of fibered states, the transition writes:

$$\begin{bmatrix} |\mathbf{F}, \underline{\mathbf{F}}, G\rangle_{[\mathbf{S}]} \\ \updownarrow \\ |[\mathbf{S}, \underline{\mathbf{S}}, G]\rangle \end{bmatrix} \rightarrow \begin{bmatrix} |\mathbf{F}', \underline{\mathbf{F}}', G\rangle_{[\mathbf{S}']} \\ \updownarrow \\ |[\mathbf{S}', \underline{\mathbf{S}}', G]\rangle \end{bmatrix} \tag{82}$$

Note that this notation also encompasses the transtion initiated by the fiber, through the mechanism:

$$|\mathbf{F}, \underline{\mathbf{F}}, G\rangle \rightarrow |\mathbf{F}', \underline{\mathbf{F}}', G\rangle \tag{83}$$

followed by basis transition:

$$|[\mathbf{S}, \underline{\mathbf{S}}, G]\rangle \rightarrow |[\mathbf{S}', \underline{\mathbf{S}}', G]\rangle$$

By including both possibilities, notation (82) underscores the fact that transitions are actually characteristics of the whole state, seen as the combination of basis and fiber states.

With modification of the group The transition towards default:

$$|[\mathbf{S}, \underline{\mathbf{S}}, G]\rangle \rightarrow \left|\left[\mathbf{S}', \underline{\mathbf{S}}^{DF}, G/DF\right]\right\rangle$$

may also impact the whole fibered state by:

$$\begin{bmatrix} |\mathbf{F}, \underline{\mathbf{F}}, G\rangle_{[\mathbf{S}]} \\ \updownarrow \\ |[\mathbf{S}, \underline{\mathbf{S}}, G]\rangle \end{bmatrix} \rightarrow \begin{bmatrix} \left|\mathbf{F}', \underline{\mathbf{F}}^{DF}, G\right\rangle_{[\mathbf{S}']} \\ \updownarrow \\ \left|\left[\mathbf{S}', \underline{\mathbf{S}}^{DF}, G/DF\right]\right\rangle \end{bmatrix} \tag{84}$$

As in the previous paragraph, this notation also encompasses the possibility of a transition towards default starting with the fiber state.

Multiple independent groups When multiple groups are independent, the transition mechanisms described for a single group, without and with modification, (82) and (84) respectively, apply to each group, so that the transition without groups modifications writes:

$$\prod_G \begin{bmatrix} |\mathbf{F}, \underline{\mathbf{F}}, G\rangle_{[\mathbf{S}]} \\ \updownarrow \\ |[\mathbf{S}, \underline{\mathbf{S}}, G]\rangle \end{bmatrix} \rightarrow \prod_G \begin{bmatrix} |\mathbf{F}', \underline{\mathbf{F}}', G\rangle_{[\mathbf{S}']} \\ \updownarrow \\ |[\mathbf{S}', \underline{\mathbf{S}}', G]\rangle \end{bmatrix} \tag{85}$$

and the transition with group modification writes:

$$\prod_G \begin{bmatrix} |\mathbf{F}, \underline{\mathbf{F}}, G\rangle_{[\mathbf{S}]} \\ \updownarrow \\ |[\mathbf{S}, \underline{\mathbf{S}}, G]\rangle \end{bmatrix} \rightarrow \prod_G \begin{bmatrix} \left|\mathbf{F}', \underline{\mathbf{F}}^{DF}, G\right\rangle'_{[\mathbf{S}']} \\ \updownarrow \\ \left|\left[\mathbf{S}', \underline{\mathbf{S}}^{DF}, G/DF\right]\right\rangle \end{bmatrix} \tag{86}$$

For multiple groups, the mechanisms of mergers and dissociations also apply, and can be written in a compact form:

$$\prod_G \begin{bmatrix} |\mathbf{F},\underline{\mathbf{F}},G\rangle_{[\mathbf{S}]} \\ \updownarrow \\ |[\mathbf{S},\underline{\mathbf{S}},G]\rangle \end{bmatrix} \rightarrow \prod_{G'} \begin{bmatrix} \left|\mathbf{F}',\underline{\mathbf{F}}',G'\right\rangle_{[\mathbf{S}']} \\ \updownarrow \\ \left|[\mathbf{S}',\underline{\mathbf{S}}',G']\right\rangle \end{bmatrix} \tag{87}$$

where the products over the groups include the possible increase (dissociation) or decrease (merger) in the number of groups.

Multiple interacting groups Among interacting groups, the various transition mechanisms can be gathered in generalization of the transition of mutliple independent groups (87), in which clusters and transitions in fiber states are included:

$$\begin{bmatrix} \prod_G |\mathbf{F},\underline{\mathbf{F}},G\rangle_{[\mathbf{S},\{G,G'\}]} \\ \updownarrow \\ \overbrace{\prod_G |[\mathbf{S},\underline{\mathbf{S}},G]\rangle}^{\mathbf{S}_{G,G'}} \end{bmatrix} \rightarrow \begin{bmatrix} \prod_G |\mathbf{F},\underline{\mathbf{F}},G\rangle'_{[\mathbf{S}',\{G,G'\}]} \\ \updownarrow \\ \overbrace{\prod_{G'} \left|[\mathbf{S}',\underline{\mathbf{S}}',G']\right\rangle}^{\mathbf{S}'_{G',G''}} \end{bmatrix}$$

This generalization encompasses transitions in fiber and basis states, so that groups phases and fiber states can experience simultaneous transitions. These mechanisms can be explored in detail by considering a sequence of successive transitions.

Sequences of transitions Basis fluctuations or transitions may induce fiber transitions, but fiber transitions may also induce basis transitions. These reciprocal interactions may trigger the following sequence of transitions:

$$\begin{bmatrix} \prod_G |\mathbf{F},\underline{\mathbf{F}},G\rangle_{[\mathbf{S},\{G,G'\}]} \\ \updownarrow \\ \overbrace{\prod_G |[\mathbf{S},\underline{\mathbf{S}},G]\rangle}^{\mathbf{S}_{G,G'}} \end{bmatrix} \rightarrow \begin{bmatrix} \prod_G |\mathbf{F},\underline{\mathbf{F}},G\rangle'_{[\mathbf{S}',\{G,G'\}]} \\ \updownarrow \\ \overbrace{\prod_G |[\mathbf{S}',\underline{\mathbf{S}},G]\rangle}^{\mathbf{S}_{G,G'}} \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \prod_G |\mathbf{F},\underline{\mathbf{F}},G\rangle'_{[\mathbf{S}',\{G,G'\}]} \\ \updownarrow \\ \overbrace{\prod_G \left|[\mathbf{S}',\underline{\mathbf{S}}',G]\right\rangle}^{\mathbf{S}_{G,G'}} \end{bmatrix} \rightarrow \ldots$$

In the above, a fluctuation in the basis state induces a first transition on the fiber state. This transition in the fiber state induces in turn a transition in the basis state, and so on... The changes in stakes induced by these several successive transitions may drive the system into an instability regime further triggering a default state, so that a transition:

$$|\mathbf{F},\underline{\mathbf{F}}\rangle' \rightarrow |\mathbf{F},\underline{\mathbf{F}}\rangle^{DF} \tag{88}$$

may occur. This default will itself impact the basis state, which will in turn enter a default state. This mecanism will emerge in the sequence of transitions through the appearance of the following term:

$$\left[\begin{array}{c}\prod_G |\mathbf{F},\underline{\mathbf{F}}\rangle^{DF}_{[\mathbf{S}',\{G,G'\}]}\\ \updownarrow\\ \overbrace{\prod_G \left|[\mathbf{S}',\underline{\mathbf{S}}',G]\right\rangle^{DF}}^{\mathbf{S}_{G,G'}}\end{array}\right]$$

10.3.2 Probabilities of transitions for fibered states

Principle The *probabilities of transition* for fibered states are similar to that of the basis or fiber states. Considering an initial state:

$$|In\rangle = \left[\begin{array}{c}\prod_G |\mathbf{F},\underline{\mathbf{F}},G\rangle_{[\mathbf{S},\{G,G'\}]}\\ \updownarrow\\ \overbrace{\prod_G |[\mathbf{S},\underline{\mathbf{S}},G]\rangle}^{\mathbf{S}_{G,G'}}\end{array}\right]$$

the transition towards a final state:

$$|Fn\rangle = \left[\begin{array}{c}\prod_G |\mathbf{F},\underline{\mathbf{F}},G\rangle'_{[\mathbf{S}',\{G,G'\}]}\\ \updownarrow\\ \overbrace{\prod_{G'} \left|[\mathbf{S}',\underline{\mathbf{S}}',G']\right\rangle}^{\mathbf{S}_{G,G'}}\end{array}\right]$$

is written formally as:

$$\langle Fn|\exp\left(-S^{B,F}\right)|In\rangle \tag{89}$$

with the action functional $S^{B,F}$ for the fibered fields is given by (68).

Derivation To derive the probabilities of transition of the fibered states (89), we must extend the definition of each field by including a time variable in its arguments. The probabilities are then computed by the following integral:

$$\begin{aligned}&\langle Fn|\exp\left(-S^{B,F}\right)|In\rangle \\ =& \int_{\substack{\left(\Lambda_{-\infty},\Lambda^{\dagger}_{-\infty}\right)=\prod_G|[\mathbf{S},\underline{\mathbf{S}},G]\rangle|\mathbf{F},\underline{\mathbf{F}},G\rangle\\ \left(\Lambda_{-\infty},\Lambda^{\dagger}_{-\infty}\right)=\prod_{G'}\left|[\mathbf{S}',\underline{\mathbf{S}}',G']\right\rangle|\mathbf{F}',\underline{\mathbf{F}}',G\rangle}} \exp\left(-\int S^{B,F}dt\right) D\left(\Lambda_{-t},\Lambda^{\dagger}_{-t}\right)\end{aligned} \tag{90}$$

where:

$$\left(\Lambda_{-t},\Lambda^{\dagger}_{-t}\right)$$

represent the time extension of the whole set of fields involved during the transition:

$$\left(\Lambda_{-t},\Lambda^{\dagger}_{-t}\right)=\left\{\left(\Lambda_t,\Lambda^{\dagger}_t,\mathbf{\Lambda}_t,\mathbf{\Lambda}^{\dagger}_t,\hat{\Psi}_t,\hat{\Psi}^{\dagger}_t,\bar{\Psi}_t,\bar{\Psi}^{\dagger}_t\right)\right\}_{\prod G,\prod G'}$$

Remark For independent basis and fiber states, this probability would factor as:

$$\overbrace{\prod_{G'}\left\langle\left[\mathbf{S}',\underline{\mathbf{S}}',G'\right]\right|}^{\mathbf{s}_{G,G'}}\exp\left(-S_B\right)\overbrace{\prod_{G}\left|\left[\mathbf{S},\underline{\mathbf{S}},G\right]\right\rangle}^{\mathbf{s}_{G,G'}}$$
$$\times\prod_{G}\ _{\left[\prod_G\left[\mathbf{S}',G\right],\prod_{G,G'}\mathbf{s}'_{G,G'}\right]}\left\langle\mathbf{F}',\underline{\mathbf{F}}'\right|\exp\left(-S_F\right)\prod_{G}\left|\mathbf{F},\underline{\mathbf{F}}\right\rangle_{\left[\mathbf{S},\{G,G'\}\right]}$$

where:

$$S_B=\sum_G S\left(\Lambda\left(\mathbf{S},\underline{\mathbf{S}}\right),G\right)+\sum_{G,G'}S_{G,G'}\left(\left(\Lambda\left(\mathbf{S}^{(G/G')},\underline{\mathbf{S}}^{(G/G')}\right)\right)\right)$$

and:

$$\begin{aligned}S_F &= \sum_G\hat{S}_{\left[\mathbf{S},\{G,G'\}\right]}\left(\hat{\Psi}\left(\hat{K},X\right)\right)+\sum_G\bar{S}_{\left[\mathbf{S},\{G,G'\}\right]}\left(\hat{\Psi}\left(\hat{K},X\right),\bar{\Psi}\left(\bar{K},X\right)\right)\\&+\sum_G\mathbf{\Lambda}^{\dagger}\left(\mathbf{F},\underline{\mathbf{F}},G\right)\\&\times\left(\frac{\delta^2\left(\hat{S}\left(\hat{\Psi}\right)+\bar{S}\left(\bar{\Psi}\right)\right)}{\delta\hat{\Psi}\left(\hat{K},X\right)\delta\hat{\Psi}^{\dagger}\left(\hat{K},X\right)}+\frac{\delta^2\bar{S}\left(\bar{\Psi}\right)\left(\hat{S}\left(\hat{\Psi}\right)+\bar{S}\left(\bar{\Psi}\right)\right)}{\delta\bar{\Psi}\left(\bar{K},X\right)\delta\bar{\Psi}^{\dagger}\left(\bar{K},X\right)}\right)\mathbf{\Lambda}\left(\mathbf{F},\underline{\mathbf{F}},G\right)\end{aligned}$$

However the interaction terms:

$$\sum_{G,G'}\delta S_{G,G'}\left(\left(\hat{\Psi}^{(G)}\left(\hat{K},X\right),\bar{\Psi}^{(G)}\left(\bar{K},X\right)\right),\left(\hat{\Psi}^{(G')}\left(\hat{K},X\right),\bar{\Psi}^{(G')}\left(\bar{K},X\right)\right)\right)$$

intertwin the two probabilities that are not independent.

We have described the various properties that a field theory for collective states should encompass. The next section synthezises the results.

11 Synthesis

To describe a complete field-theoretic formalism, we reinterpreted collective states as collections of interacting sub-collective states, namely groups and their associated phases, and differenciated the variables of these sub-collective states between basis and fiber variables. Viewed in this light, the dynamics of collective states evolve along two complementary and interacting directions: that of the basis and that of the fiber collective states. For each, we provided specific notations, to account for deviations around the static collective states.

The dynamics of the basis states and of their transitions is described by an infinite set of basis deviation fields, each associated with a possible equilibrium $\underline{\mathbf{S}}$ of the basis variables:

$$\Lambda\left(\mathbf{S},\underline{\mathbf{S}},G\right)$$

and their realizations, the basis deviation states:

$$\left|\left[\mathbf{S},\underline{\mathbf{S}},G\right]\right\rangle$$

The dynamics of the fiber states and of their transitions is described by the fiber equilibrium fields describing the with multiple equilibria of the fiber state. These fields are denoted:

$$\hat{\Psi},\bar{\Psi}$$

These fiber equilibrium fields are completed by the infinite set of fiber deviation fields, one for each value of the fiber equilibrium $\underline{\mathbf{F}}$:

$$\mathbf{\Lambda}\left(\mathbf{F},\underline{\mathbf{F}},G\right)$$

The realizations of this set of fields are the fiber deviation states given by:

$$\left|\mathbf{F},\underline{\mathbf{F}},G\right\rangle \tag{91}$$

Ultimately, the combinations of basis and fiber deviation states, the so-called fibered deviation states, have their own dynamics, which emerge from the interactions between basis and fiber deviation fields. These fibered deviation states, denoted as:

$$\left[\begin{array}{c}\prod_G \left|\mathbf{F},\underline{\mathbf{F}},G\right\rangle_{[\mathbf{S},\{G,G'\}]} \\ \updownarrow \\ \overbrace{\mathbf{S}_{G,G'}\prod_G \left|[\mathbf{S},\underline{\mathbf{S}},G]\right\rangle}\end{array}\right]$$

describe the system as a whole and are the realizations of the interactions between basis deviation fields and fiber equilibrium and deviation fields.

The dynamics of these fibered deviation states, of their interactions and of their transitions are formulated through three action functionals S_B, S_F and $S^{B,F}$ for the basis, fiber, and fibered states, respectively. The action functionals S_B and S_F describe the internal dynamics of their respective deviations states, and the internal interactions among their sub-collective states, whereas the action functional $S^{B,F}$ of the fibered fields is the sum of the action functionals S_B and S_F, to which some terms have been added to incorporate the interactions between the basis and fiber fields.

Using this unified framework, we can determine the transition probabilities between any initial and final fibered deviation state of the system, denoted $|In\rangle$ and $|Fn\rangle$ respectively, and given by:

$$\langle Fn|\exp\left(-S^{B,F}\right)|In\rangle$$

Since the action functional computes the transition probabilities between initial and final fibered deviation states, we can also interpret its terms to inspect the various transition mechanisms of the fibered deviation states, as well as their possible sequences. Moreover, the action functional of the fibered fields also allows to detect the most stable sub-collective deviation states of a system, and analyze the internal reorganization of a collective state.

However, this general structure of the various action functionals of the system does not specify the precise forms that interactions between groups, or between basis and fiber components, should take. It is the detailed study of these interaction structures that will determine the mechanisms governing state transitions and enable the explicit computation of transition probabilities. Such an analysis constitutes the next stage of this research.

12 Conclusion

In this work, we have summarized the main properties of collective states in systems with a large number of agents, firms, investors, and banks, and we have detailed the analysis of the stability and instability of such states. In order to account for the possibility of unstable fluctuations of these collective states, inducing transitions between them, we were led to consider the necessity of an extension of our formalism aimed at describing collective states in their entirety, in a dynamical and interacting framework. We have identified the essential elements of such an extension: an infinity of deviation fields representing deviation states around collective states; the need to distinguish these fields for basis states and fiber states; a typology of the different possible types of transitions between collective states; and the general form of the action functionals for deviation fields inducing these transitions. On the basis of this framework, the detailed development of the formalism and its applications will be presented in the fourth part of this series.

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Appendix 1

We present a summary of the models developped in Gosselin and Lotz (2024, 2025,a,b). Some technical details are given in the subsequent Appendices.

A1.1 Fields and action functionals

Let us first gather all the fields involved in the collective states in Gosselin and Lotz (2025a,b)[51]. They are the field of investors and banks, and the field of their connections, the stakes. We have described the fields of investors $\hat{\Psi}(K,X)$ and the field of banks $\bar{\Psi}(K,X)$ in Gosselin and Lotz (2024). Their argument are their respective sector and level of capital and sector.

The dynamics of the level of capital is determined by the structure of stakes between sectors, along with investors and banks' respective shares of firms' returns. This dynamics is encompassed in the fields action functionals[52]. Gosselin and Lotz (2025a,b) endogeneized their respective stakes by two fields, Γ et $\bar{\Gamma}$. In the following, we present a complete description of these fields and their action functionals, for banks, investors and stakes.

A1.1.1 Banks

The field $\bar{\Psi}$ describes the banking system. It depends on the two variables, the level of disposable capital $\bar{K}$ and the sector X. Its action functional is given by:

$$\bar{S}(\bar{\Psi}) = \int \left(-\bar{\Psi}^{\dagger}(\bar{K},X)\nabla^2\bar{\Psi}(\bar{K},X) + \left(\frac{\bar{g}^2(\bar{K},X)}{2\sigma_{\bar{K}}^2} + \frac{\bar{g}(\bar{K},X)}{2\bar{K}} \right) \left|\bar{\Psi}(\bar{K},X)\right|^2 \right) d(\bar{K},X) \tag{92}$$

where:

$$\bar{g}\left(\hat{K},X\right) = \left(1 - \bar{M}\left|\bar{\Psi}(\bar{K},X)\right|^2\right)^{-1} \bar{f}(\bar{K},X)$$

where the function $\bar{f}(\bar{K},X)$ is the return of a bank X with disposable capital $\bar{K}$, and the function $\bar{g}\left(\hat{K},X\right)$ is an induced return that accounts for the propagation of the return among agents, the diffusion being realized[53] through the matrix M.

The action (92) describes the accumulation of capital $\bar{K}$ by banks in a sector X. The squared field amplitude $\left|\bar{\Psi}(\bar{K},X)\right|^2$ measures the density of agents holding capital $\bar{K}$ in sector X. By determining the fields $\bar{\Psi}(\bar{K},X)$ that minimize (92), one obtains the most probable distributions of capital among banks across the different sectors, as well as the average level of accumulated capital in each sector.

A1.1.2 Investors

The field $\hat{\Psi}$ describes the network of investors. It depends on the level of disposable capital $\hat{K}$ and the sector X. Its action functional is given by:

$$\hat{S}\left(\hat{\Psi}\right) = \int \left(-\hat{\Psi}^{\dagger}\left(\hat{K},X\right)\nabla^2\hat{\Psi}\left(\hat{K},X\right) + \left(\frac{\hat{g}^2\left(\hat{K},X\right)}{2\sigma_{\hat{K}}^2} + \frac{\hat{g}\left(\hat{K},X\right)}{2\hat{K}} \right) \left|\hat{\Psi}\left(\hat{K},X\right)\right|^2 \right) d\left(\hat{K},X\right) \tag{93}$$

[51] Some details about the microfundations and their fields translations are given in Appendix 2.

[52] A third field $\Psi(K,X)$ describing the firms was considered in Gosselin and Lotz (2024). In Gosselin and Lotz (2025) and in this work the relevant firms quantities were directly integrated in the equations for investors and banks

[53] See appendix 2.

where we define the induced return $\hat{g}$ as:

$$\hat{g}\left(\hat{K},X\right)=\left(1-\hat{M}\left|\hat{\Psi}\left(\hat{K},X\right)\right|^{2}\right)^{-1}\hat{f}\left(\hat{K},X\right)+\left(1-\hat{M}\right)^{-1}N\left(1-\bar{M}\right)^{-1}\bar{f}\left(\bar{K},X\right)$$

This formula describes how the returns of investors, $\hat{f}\left(\hat{K},X\right)$, and of banks $\bar{f}\left(\bar{K},X\right)$ are diffused to other investors through the matrices $\bar{M}$, $\hat{M}$ and N[54]. Both formula (92) and (93) depend on the network of connections between agents, that is, on their respective stakes. The minimization of this action functional will be constrained by the return equation for investors.

The action (93) describes the accumulation of capital $\hat{K}$ by investors in a sector X. The squared field amplitude $\left|\hat{\Psi}\left(\hat{K},X\right)\right|^{2}$ measures the density of agents holding capital $\hat{K}$ in sector X. By determining the fields $\bar{\Psi}\left(\bar{K},X\right)$ that minimize (93), one obtains the most probable distributions of capital among investors across the different sectors, as well as the average level of accumulated capital in each sector.

A1.1.3 Firms

Similarly the action functional for firms

$$-\Psi^{\dagger}\left(K,X\right)\left(\nabla_{K_{p}}\left(\sigma_{K}^{2}\nabla_{K_{p}}-f_{1}'\left(K,X\right)K_{p}\right)\right)\Psi\left(K,X\right)+\frac{1}{2\epsilon}\left(\left|\Psi\left(K,X\right)\right|^{2}-\left|\Psi_{0}\left(X\right)\right|^{2}\right)^{2} \tag{94}$$

where:

$$\begin{aligned} f_{1}'\left(X\right) &= \left(1+\underline{k}_{2}\left(X\right)\right)f_{1}\left(X\right)-\bar{r}\underline{k}_{2}\left(X\right) \\ &= f_{1}\left(X\right)+\left(f_{1}\left(X\right)-\bar{r}\right)\int k_{2}\left(X,X_{1}\right)\hat{K}_{1}\left|\hat{\Psi}\left(\hat{K}_{1},X_{1}\right)\right|^{2}d\hat{K}_{1}dX_{1} \end{aligned}$$

computes the firms returns, net from loans repaiements

A1.2 Stakes

The connections between investors and banks have been endogeneized in Gosselin and Lotz (2025a,b). They are described by two fields, one for the stakes owned by investors, the other for the stakes owned by banks, whose variables are the banks and investors' stakes respectively, along with the positions in the sector space.

12.0.1 A1.2.1 Investors

The field of investors' stakes The field of investors writes:

$$\Gamma\left(\hat{S}^{(T)},X',X\right)\equiv\Gamma\left(S_{1},\hat{S}_{1},S_{2},\hat{S}_{2},X',X\right)$$

where the four possible types of stakes are gathered in a vector $\hat{S}^{(T)}$:

$$\hat{S}^{(T)}=\left(S_{1},\hat{S}_{1},S_{2},\hat{S}_{2}\right)$$

in which the arguments S_1 and S_2 are the shares and loans taken by investor X in his sector firms, respectively, while $\hat{S}_1$ and $\hat{S}_2$ are those taken in an investor X'. This field describes the possible repatitions of investors investments' across sectors.

[54] The field definition of these matrices $\bar{M}$, $\hat{M}$ and N are given in Appendix 2.

The action functional The action functionals $S(\Gamma)$ for the field $\Gamma\left(\hat{S}^{(T)},X',X\right)$ is[55]:

$$\begin{aligned}
S(\Gamma) \quad = \quad & -\int \sigma_{\hat{K}}^{2}\Gamma^{\dagger}\left(\hat{S}^{(T)},X',X\right)\nabla_{\hat{S}_{\eta}^{(T)}}^{2}\Gamma\left(\hat{S}^{(T)},X',X\right)d\left(\hat{S}^{(T)},X',X\right) \qquad (95)\\
& -\int \beta\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^{2}d\left(\hat{S}^{(T)},X',X\right)\\
& +\sum_{\eta}\int\left(\frac{\left(\hat{S}_{\eta}^{(T)}\right)^{2}}{2\hat{w}_{\eta}^{T}(X',X)}-\hat{V}_{\eta}\hat{S}_{\eta}^{(T)}-\beta\right)\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^{2}d\left(\hat{S}^{(T)},X',X\right)\\
& +\int\lambda(X)\left(\sum_{\eta}\int\hat{S}_{\eta}^{(T)}\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^{2}dX'd\hat{S}^{(T)}-1\right)\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^{2}d\left(\hat{S}^{(T)},X',X\right)
\end{aligned}$$

where the components of the vector $\left(\hat{w}_{\eta}^{T}(X',X)\right)$ are the field-translated (inverse) uncertainty coefficients $\hat{w}_{\eta ij}$ and $w_{\eta ik}$:

$$\left(\hat{w}_{\eta}^{T}(X',X)\right)=\left(\hat{w}_{E}(X',X),\hat{w}_{L}(X',X),w_{E}(X,X),w_{L}(X,X)\right)$$

which reflect the uncertainty perceived by investors X about their stakes (either equity or loans) in investors X' and in firms X. The functions $\hat{V}_{\eta}\left(\hat{S}_{\eta}\right)$ involved in (95) are defined in Appendix 2.

The Lagrange multipliers $\lambda(X)$ implement the constraint:

$$\sum_{\eta}\int\hat{S}_{\eta}^{(T)}\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^{2}dX'd\hat{S}^{(T)}=1$$

where the integral accounts for the sum of stakes of an investor X investing in various sectors X', and the factor $\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^{2}$ weighs the stakes $\hat{S}_{\eta}^{(T)}$ by the number of investors X' benefiting from these stakes.

We define the partial averages:

$$\begin{aligned}
S_{\eta}(X,X) &= \int S_{\eta}\left|\Gamma\left(S_{E},\hat{S}_{E},S_{L},\hat{S}_{L},X',X\right)\right|^{2}d\left(S_{E},\hat{S}_{E},S_{L},\hat{S}_{L},X'\right)\\
\hat{S}_{\eta}(X',X) &= \int\hat{S}_{\eta}\left|\Gamma\left(S_{E},\hat{S}_{E},S_{L},\hat{S}_{L},X',X\right)\right|^{2}d\left(S_{E},\hat{S}_{E},S_{L},\hat{S}_{L}\right)
\end{aligned}$$

so that the constraint on allocation writes[56]:

$$\int\left(\hat{S}_{E}(X',X)+\hat{S}_{L}(X',X)\right)dX'+\int\left(S_{E}(X',X)+S_{L}(X',X)\right)dX'=1 \qquad (96)$$

A1.2.2 Banks

The field of banks' stakes The field of banks is defined as:

$$\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)=\bar{\Gamma}\left(\bar{S}_{E},S_{E}^{B},\hat{S}_{E}^{B},\bar{S}_{L},S_{L}^{B},\hat{S}_{L}^{B},X',X\right)$$

[55] Derived in Gosselin and Lotz (2025a,b).

[56] Appendix 2 in Gosselin and Lotz (2025) derives the minimization equation of (95) in terms of the sectoral averages:

$$\hat{S}_{\eta}^{(T)}\left(\hat{X}',\hat{X}\right)=\frac{\int\hat{S}_{\eta}^{(T)}\left|\Gamma\left(\hat{S}^{(T)},\hat{X}',\hat{X}\right)\right|^{2}d\hat{S}^{(T)}}{\int\left|\Gamma\left(\hat{S}^{(T)},\hat{X}',\hat{X}\right)\right|^{2}d\hat{S}^{(T)}}$$

with:

$$\bar{S}^{(T)}=\left(\bar{S}_E,S_E^B,\hat{S}_E^B,\bar{S}_L,S_L^B,\hat{S}_L^B\right)$$

The arguments S_E^B and S_L^B are the stakes taken by a bank X in firms of the same sector, through shares and loans respectively, while $\hat{S}_E^B$ and S_L^B are those taken in an investor X', and $\bar{S}_E$ and $\bar{S}_L$ are the stakes taken in a bank X'. As for investors, this field describes the possible repartitions of banks investments' across sectors.

The action functional The action functional $S\left(\bar{\Gamma}\right)$ for the field $\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)$ of banks'stakes is defined as[57]:

$$\begin{aligned}
S\left(\bar{\Gamma}\right) &= -\sigma_{\hat{K}}^2\sum_{\eta}\int\bar{\Gamma}^{\dagger}\left(\bar{S}^{(T)},X',X\right)\nabla_{\bar{S}_{\eta}^{(T)}}^2\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)d\left(\bar{S}^{(T)},X',X\right) \\
&+\int\left(\sum_{\eta}\left(\frac{\left(\bar{S}_{\eta}^{(T)}\right)^2}{2\bar{w}_{\eta}\left(X',X\right)}-\bar{V}_{\eta}\bar{S}_{\eta}^{(T)}\right)-\beta\right)\left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^2 d\left(\bar{S}^{(T)},X',X\right) \\
&+\int\lambda\left(X\right)\left(\sum_{\eta=E,L}\int\bar{S}_{\eta}^B\left(X',X\right)dX'+\int\hat{S}_E^B\left(X',X\right)dX'+S_E^B\left(X,X\right)-1\right)\left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^2 \\
&+\int\lambda'\left(X\right)\left(\int\hat{S}_L^B\left(X',X\right)dX'+S_L^B\left(X,X\right)-\kappa\left(1-\bar{S}_{\eta}^B\left(X\right)\right)\right)\left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^2
\end{aligned}\tag{97}$$

where we defined the various averages in the states defined by the fields:

$$\begin{aligned}
\bar{S}_{\eta}\left(X',X\right) &= \int\bar{S}_{\eta}\left|\bar{\Gamma}\right|^2 d\left(\bar{S}^{(T)},X'\right) \\
S_{\eta}^B\left(X,X\right) &= \int S_{\eta}^B\left|\bar{\Gamma}\right|^2 d\left(\bar{S}^{(T)},X'\right) \\
\hat{S}_{\eta}^B\left(X',X\right) &= \int\hat{S}_{\eta}^B\left|\bar{\Gamma}\right|^2 d\left(\bar{S}^{(T)},X'\right)
\end{aligned}$$

and the coefficients:

$$\begin{gathered}
\bar{w}_E^T\left(X',X\right)=\bar{w}_E\left(X',X\right),\ \bar{w}_L^T\left(X',X\right)=\bar{w}_L\left(X',X\right) \\
\hat{w}_3^T\left(X',X\right)=\hat{w}_E\left(X',X\right),\ \bar{w}_4^T\left(X',X\right)=\hat{w}_L\left(X',X\right) \\
\hat{w}_5^T\left(X',X\right)=w_E\left(X,X\right),\ \bar{w}_6^T\left(X',X\right)=w_L\left(X,X\right)
\end{gathered}$$

are the field translation of the inter- and intra-sectoral uncertainty coefficients.

The Lagrange multipliers $\lambda\left(X\right)$ and $\lambda'\left(X\right)$ implement the constraints on allocations:

$$\sum_{\eta}\int\bar{S}_{\eta}^B\left(X',X\right)dX'+\int\hat{S}_E^B\left(X',X\right)dX'+S_E^B\left(X,X\right)=1\tag{98}$$

$$\int\hat{S}_L^B\left(X',X\right)dX'+S_L^B\left(X,X\right)=\kappa\left(1-\bar{S}_{\eta}^B\left(X\right)\right)$$

$$\bar{S}^{(T)}\left(X',X\right)\left|\Gamma\left(X',X\right)\right|^2=\int\bar{S}^{(T)}\left|\Gamma\left(\bar{S}^{(T)},X',X\right)\right|^2$$

The functions $\bar{V}_{\eta}\left(\bar{S}\right)$, involved in the expression of $S\left(\bar{\Gamma}\right)$ are defined in Appendix 2.

[57] See Appendix 1.

Appendix 1.3 Minimization equations of action functionals

The collective states of the system are defined by minimization equations of each field of the system. We first consider the minimization equations for (92) and (93) leading to the field return equations for bank investors and banks[58], and then the minimization equations for fields of stakes, leading to find the stakes defining the collective state.

A.1.3.1 Firms, investors and banks

A.1.3.1.1 Field equations The minimization for investors and banks fields are obtained by minimizing:

$$\bar{S}\left(\bar{\Psi}\right)+\hat{S}\left(\hat{\Psi}\right)$$

leading to the two equations:

$$\frac{\delta}{\delta\bar{\Psi}\left(\bar{K},X\right)}\left(\bar{S}\left(\bar{\Psi}\right)+\hat{S}\left(\hat{\Psi}\right)\right)=0$$

for banks, and:

$$\frac{\delta}{\delta\hat{\Psi}\left(\hat{K},X\right)}\left(\bar{S}\left(\bar{\Psi}\right)+\hat{S}\left(\hat{\Psi}\right)\right)=0$$

for investors. These equations write:

$$\begin{aligned}0=&\frac{\hat{K}_1^2\hat{g}^2\left(\hat{K}_1,X_1\right)}{2\sigma_{\hat{K}}^2}+\frac{\hat{g}\left(\hat{K}_1,X_1\right)}{2}\\&+\int\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2\left(\frac{\hat{K}^2\hat{g}\left(\hat{K},X,\Psi,\hat{\Psi}\right)}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\frac{\delta\hat{g}\left(\hat{K},X\right)}{\delta\left|\hat{\Psi}\left(\hat{K}_1,X_1\right)\right|^2}+\frac{1}{\hat{\mu}}\left(\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2-\left|\hat{\Psi}_0\left(X\right)\right|^2\right)\end{aligned}\tag{99}$$

$$\begin{aligned}0=&\left(\frac{\bar{K}_1^2\bar{g}^2\left(\bar{K}_1,X_1\right)}{\sigma_{\hat{K}}^2}+\frac{\bar{g}\left(\bar{K}_1,X_1\right)}{2}\right)\\&+\int\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2\left(\frac{\hat{K}^2\hat{g}\left(\hat{K},X\right)}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\frac{\delta\hat{g}\left(\hat{K},X\right)}{\delta\left|\bar{\Psi}\left(\bar{K}_1,X_1\right)\right|^2}+\frac{1}{\bar{\mu}}\left(\left|\bar{\Psi}\left(\bar{K}_1,X_1\right)\right|^2-\left|\bar{\Psi}_0\left(X_1\right)\right|^2\right)\end{aligned}\tag{100}$$

This yields the backgrounds fields $\bar{\Psi}$ and $\hat{\Psi}$ under the constraint by the return equations:

These non-local equations link the distribution of capital of investors and banks in one sector, measured by $\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2$ and $\left|\bar{\Psi}\left(\bar{K}_1,X_1\right)\right|^2$ respectively, to the returns of provided by the various sectors, through cross investment between banks and investors.

These equations have to be constrained by the return equations for investors and banks. These equations translate that investors and banks returns are constrained across sectors due to agents mutual investments (see Appendix 2 for details).

Similarly, the minimization equation equations, for (94) reduce to:

$$0=\left(\frac{f_1'^2\left(K,X\right)}{\sigma_{\hat{K}}^2}+\frac{\frac{d}{dK}f_1'\left(K,X\right)}{2}\right)+\frac{1}{\epsilon}\left(\left|\Psi\left(K,X\right)\right|^2-\left|\Psi_0\left(X\right)\right|^2\right)\tag{101}$$

[58] See Appendix 2 for some details.

A.1.3.1.2 The return equations The return equations for banks and investors constrain the previous minimization equations by linking returns of investors and banks through mutual participations and depend on the return of firms in the various sectors[59].

Assuming[60] investors and banks invest in neighbouring firms, we can write:

$$\begin{aligned} S_E(X',X) &= S_E(X,X)\,\delta(X'-X) \\ S_L(X',X) &= S_L(X,X)\,\delta(X'-X) \end{aligned}$$

$$\begin{aligned} S_E^B(X',X) &= S_E^B(X,X)\,\delta(X'-X) \\ S_L^B(X',X) &= S_L^B(X,X)\,\delta(X'-X) \end{aligned}$$

where $\delta(X'-X)$ is the Dirac function. It is equal to 0 for $X' \neq X$. Under these assumptions, the constraint (96) simplifies to:

$$\int \left(\hat{S}_E(X',X) + \hat{S}_L(X',X)\right) dX' + S_E(X,X) + S_L(X,X) = 1$$

and the constraint (98) simplifies as:

$$\int \left(\bar{S}_E(X',X) + \bar{S}_L(X',X)\right) dX' + \int \hat{S}_E^B(X',X)\, dX' + S_E^B(X,X) = 1$$

We showed in Gosselin and Lotz (2025a,b) that the return equation for investors writes[61]:

$$0 = \int \left(\delta(X'-X) - \hat{S}_E(X',X)\right) \widehat{DF}(X')\,\hat{R}_{exc}(X')\, dX' - S_E(X,X)\, R_{exc}(X) \tag{102}$$

where $\widehat{DF}(X)$ is the investor's discount factor for debt repayment and writes[62]:

$$\widehat{DF}(X) = \frac{1-\hat{S}(X)}{1-\hat{S}_E(X)}$$

and:

$$\hat{R}_{exc}(X') = f(X') - \bar{r}$$

$$R_{exc}(X) = f(X) - \bar{r}$$

are the investors and firms excess returns respectively.

Similarly, the banks return equation without default for banks writes:

$$\begin{aligned} 0 \quad = \quad & \int \left(\delta(X'-X) - \bar{S}_E(X',X)\right) \overline{DF}(X')\,\bar{R}_{exc}(X') \\ & - \int \hat{S}_E^B(X',X)\,\widehat{DF}(X')\,\hat{R}_{exc}(X') - S_E^B(X',X)\, R_{exc}(X) \end{aligned} \tag{103}$$

[59] More details are given in Gosselin and Lotz (2025a,b). Appendix 2 presents these returns equations as constraints.

[60] As in Gosselin and Lotz (2024).

[61] See Appendix 1 for details. The case of no default, or default are studied in Appendices 1.2, 1.3.

[62] In the following, we will also encounter the firm's discount factor for debt repayment:

$$DF(X) = \frac{1-S(X)}{1-S_E(X)}$$

where $\overline{DF}(X)$ is the bank's discount factor for debt repayment and writes:

$$\overline{DF}(X)=\frac{1-\bar{S}(X)}{1-\bar{S}_E(X)}$$

The excess returns $\bar{R}_{exc}(X')$ of banks is given by:

$$\bar{R}_{exc}(X')=\bar{f}(X')-(1+\kappa)\bar{r}$$

The full equations with default[63] will be presented later in the text. Note that, in the following, we will also encounter the firm's discount factor for debt repayment:

$$DF(X)=\frac{1-\left(S(X)+S^B(X)\right)}{1-S_E(X)-S_E^B(X)}$$

A.1.3.2 Stakes field equations

The collective states correspond to the solution of the minimization equations for the stakes' investors field $\Gamma\left(\hat{S}^{(T)},X',X\right)$ and $\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)$:

$$\begin{aligned}\frac{\delta S(\Gamma)}{\delta\Gamma\left(\hat{S}^{(T)},X',X\right)}&=0\\ \frac{\delta S(\bar{\Gamma})}{\delta\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)}&=0\end{aligned}$$

which write:

$$0=\left(\sum_\eta\left(-\sigma_{\hat{K}}^2\nabla_{\hat{S}_\eta^{(T)}}^2+\frac{\left(\hat{S}_\eta^{(T)}\right)^2}{2\hat{w}_\eta}-\hat{V}_\eta\hat{S}_\eta^{(T)}+\lambda(X)\left\Vert\Gamma\left(\hat{S}^{(T)},X',X\right)\right\Vert_X^2\hat{S}_\eta^{(T)}\right)-\beta\right)\Gamma\left(\hat{S}^{(T)},X',X\right)\tag{104}$$

for investors stakes fields, with $\lambda(X)$ a Lagrange multiplier implementing the constraint (96) on stakes, and:

$$\begin{aligned}0=&-\sigma_{\hat{K}}^2\sum_\eta\nabla_{\bar{S}_\eta^{(T)}}^2\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\\&+\left(\sum_\eta\left(\frac{\left(\bar{S}_\eta^{(T)}\right)^2}{2\bar{w}_\eta}-\bar{V}_\eta^{(T)}\bar{S}_\eta^T+\lambda_\eta(X)\bar{S}_\eta^T\right)-\beta\right)\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\end{aligned}\tag{105}$$

for banks field of stakes $\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)$, where:

$$\begin{aligned}\lambda_\eta(X)&=\lambda(X)\text{ for }\eta=1,2,3,5\\ \lambda_\eta(X)&=\lambda'(X)\text{ for }\eta=5,6\end{aligned}$$

are Lagrange multiplier implementing the constraints (96) and (98).

[63] See Appendix 1.

A1.3.3 Equations in terms of sectoral stakes

We show in Gosselin and Lotz (2025a,b) that the minimization equations for the field of stakes $\Gamma\left(\hat{S}^{(T)}, X', X\right)$ and $\bar{\Gamma}\left(\bar{S}^{(T)}, X', X\right)$ can be rewritten in terms of sectoral stakes.

Actually, in first approximation, the solutions of (104) take the form:

$$\Gamma_{0,X',X}\left(\hat{S}^{(T)}\right) = N \exp\left(-\sum_{\eta} \frac{\left(\hat{S}_{\eta}^{(T)} - \hat{S}_{\eta}^{(T)}\left(X', X\right)\right)^2}{2\sigma_{\hat{K}}^2}\right) \tag{106}$$

where N is a normalization factor. Solving the minimization equations thus reduces to finding the stakes invested by sector X into sector X', i.e. the sectoral averages $\hat{S}_{\eta}^{(T)}\left(X', X\right)$. They satisfy[64]:

$$\hat{S}_{\eta}^{(T)}\left(X', X\right) = \hat{w}_{\eta}\left(X', X\right)\left(\hat{V}_{\eta}\left(X', X\right) + \lambda\left(X\right)\right) \tag{107}$$

and depend on the Lagrange multipliers[65].

The minimization equation for banks is similar to the investors equation, and the solutions have the form[66]:

$$\Gamma_{0,X',X}\left(\bar{S}^{(T)}\right)\Gamma\left(X', X\right)$$

where:

$$\Gamma_{0,X',X}\left(\bar{S}^{(T)}\right) = \bar{N} \exp\left(-\sum_{\eta} \frac{\left(\bar{S}_{\eta}^{(T)} - \bar{S}_{\eta}^{(T)}\left(X', X\right)\right)^2}{2\sigma_{\hat{K}}^2}\right) \tag{108}$$

where $\bar{N}$ is a normalization factor. Solving the minimization equations thus reduces to finding the stakes invested by sector X into sector X', i.e.the sectoral averages $\bar{S}_{\eta}^{(T)}\left(X', X\right)$. We show in appendix 16 of Gosselin and Lotz (2025b) that $\bar{S}_{\eta}^{(T)}\left(X', X\right)$ satisfies:

$$\bar{S}_{\eta}^{(T)}\left(X', X\right) = \bar{w}_{\eta}\left(X', X\right)\left(\bar{V}_{\eta}\left(X', X\right) + \lambda_{\eta}\left(X\right)\right) \tag{109}$$

and depend on the Lagrange multipliers $\lambda_{\eta}\left(X\right)$.

A1.3.4 The equations for uncertainty

Uncertainty is measured by the coefficients $\hat{w}$, $\bar{w}$, which are endogenous. These coefficients themselves depend on the stakes; we therefore simply recall their functional form.

A.1.3.4.1 Investors The equation that defines the weight of inverse uncertainty for cross-investment between investors, derived in Gosselin and Lotz (2025a,b), is:

$$\hat{w}\left(X', X\right) = \frac{2\left(1 - \left(\gamma\left\langle\hat{S}_E\left(X\right)\right\rangle\right)^2\right)\hat{w}_E^{(0)}\left(X', X\right)}{1 + \hat{w}_E^{(0)}\left(X', X\right)\left(1 - \left(\gamma\left\langle\hat{S}_E\left(X\right)\right\rangle\right)^2\right) + \Delta\left(\gamma\left\langle\hat{S}_E\left(X_1, X'\right)\right\rangle_{X_1}\right)^2} \tag{110}$$

The coefficient $\hat{w}_E^{(0)}\left(X', X\right)$ is a factor of local inverse uncertainty: it is a factor of confidence of investors X in investors X'. The higher this factor, the higher the investments in sector X'. As

[64]See Appendix 2 in Gosselin and Lotz (2025).
[65]See Appendix 2 in Gosselin and Lotz (2025).
[66]See Appendix 4.

such, this coefficient captures the characteristics of local uncertainty that induce deviations from average behavior. The parameter γ is the average uncertainty of the distance-dependent investment paths and writes:

$$\gamma^2 \simeq \left(\hat{w}_E^{(0)}\left(\left(X'\right)', X_{m-1}\right)...\hat{w}_E^{(0)}\left(X_1, X'\right)\right)^{-\frac{1}{m}}$$

The weight of inverse uncertainty for investments in firms is given by:

$$w\left(X, X\right) = 1 - \left\langle w\left(X', X\right)\right\rangle_{X'} \tag{111}$$

The contribution:

$$\Delta\left(\gamma\left\langle \hat{S}_E\left(X_1, X'\right)\right\rangle_{X_1}\right)^2 = \left(\gamma\left\langle \hat{S}_E\left(X_1, X'\right)\right\rangle_{X_1}\right)^2 - \left(\gamma\left\langle \hat{S}_E\left(X\right)\right\rangle\right)^2$$

in equation (110) represents the gap between the risk perception of investor X' and that of the market. It increases with the risk perception of investor X', so that the higher this gap, the lower the coefficient $\hat{w}\left(X', X\right)$, and the lower the investment of investor X' in investor X.

A.1.3.4.2 Banks Similarly the weights for inverse uncertainty of banks are:

$$\left(\bar{w}\left(X', X\right)\right)^{-1} = 1 + \frac{1}{2}\left(\frac{\overline{IRG}\left(X', X\right)}{\widehat{IR^B}\left(X', X\right)} + \frac{\overline{IRG}\left(X', X\right)}{\xi^2}\right) \tag{112}$$

and those for banks investing investors are given by:

$$\left(\hat{w}_E^B\left(X', X\right)\right)^{-1} = 1 + 2\frac{\widehat{IR^B}\left(X', X\right)}{\overline{IRG}\left(X', X\right)} + \frac{\widehat{IR^B}\left(X', X\right)}{\xi^2} \tag{113}$$

$$w_E^B\left(X', X\right) = 1 - \bar{w}\left(X', X\right) - \hat{w}_E^B\left(X', X\right) \tag{114}$$

where the various quantities are defined in Appendix 3.

Since the functions $\hat{w}$ and w both depend on the stakes, their expression complete the minimization equations and yield $\hat{S}_E\left(X', X\right)$, $\hat{S}\left(X', X\right)$, $S_E\left(X, X\right)$, $S\left(X, X\right)$.

Appendix 1.4 Resolution and Collective states: no-default scenario

A1.4.1 Principle of resolution

The collective states are defined by the solutions of (99), (100), (102), (103), (107), (109).

The resolution is in made of two blocks. First, the collective states for investors and banks are defined by the solutions of (99), (100). The resolution defines also what we will call fiber states. The fields $\hat{\Psi}\left(\hat{K}, X\right)$ and $\bar{\Psi}\left(\bar{K}, X\right)$ will describe the distribution of capital in each sector, but will be dependent on the distribution of stakes and returns across the sector space. The formula were derived in Gosselin and Lotz (2024).

Once the formulas for $\hat{\Psi}\left(\hat{K}, X\right)$ and $\bar{\Psi}\left(\bar{K}, X\right)$ are obtained, the second block of equations is given by (102), (103), (107), (109). This set of equations, is solved in several steps.

The return equations in sector X, equations (102), (103), connect all investors in the sector space, and as such comprises three types of variables: the inward aggregate stakes on investors and banks in sector X; the returns of investors X, banks X and firms X, taken independently; and the average returns of all investors of the sector space.

The resolution follows a four-step process:

1. We average the return equation over all sectors. This yields the averages of all the variables of the equation: average stakes, levels of capital and returns.

2. We then replace these results in the return equation, and further replace the aggregate stakes in terms of returns, so that the return equation depends only on returns.

3. Solving this equation yields the returns per sector and, in turn, the aggregate stakes and the disposable capital per sector.

4. Using equations (107), (109), we reconstruct the remaining quantities of the model.

A1.4.2 Form of the solution: groups and sub-collective states

To solve the full system, we must determine the levels of capital and returns of each type of agents for each sector of the sector space S, along with the distribution of stakes across all these sectors. Taken together, these quantities define a collective state of the system.

Since capital and returns can be derived from the level of stakes between investors, a collective state is described by a set of values of stakes for investors:

$$\left\{\hat{S}_E\left(X',X\right),\hat{S}\left(X',X\right),S_E\left(X,X\right),S\left(X,X\right)\right\}_{(X',X)\in S}$$

and for banks:

$$\left\{\hat{S}_E^B\left(X',X\right),\hat{S}^B\left(X',X\right),S_E^B\left(X,X\right),S^B\left(X,X\right),\bar{S}_E\left(X',X\right),\bar{S}^B\left(X',X\right)\right\}_{(X',X)\in S}$$

These quantities are governed by equations (102) and (103). These are non-local equations that involve the network of connections within each group.

We have seen that, under uncertainty, agents connect to only a finite number of neighbors, so that agents are organized into several and loosely-connected groups. Each of these groups G can be in one among several possible phase, each defined by a distribution of stakes between the agents of G:

$$\begin{array}{l}\left\{\hat{S}_E\left(X',X\right),\hat{S}\left(X',X\right),S_E\left(X,X\right),S\left(X,X\right),\right.\\ \left.\hat{S}_E^B\left(X',X\right),\hat{S}^B\left(X',X\right),S_E^B\left(X,X\right),S^B\left(X,X\right),\bar{S}_E\left(X',X\right),\bar{S}^B\left(X',X\right)\right\}_{(X',X)\in G}\end{array}$$

We will call a *sub-collective state* the combination of one group G and one of its possible phases. Thus, collective states organize into several distinct, weakly-interacting sub-collective states, so that, in first approximation, a collective state can be seen as a set of sub-collective states, that writes:

$$\cup_\alpha \quad \begin{array}{l}\left\{\hat{S}_E\left(X',X\right),\hat{S}\left(X',X\right),S_E\left(X,X\right),S\left(X,X\right),\right.\\ \left.\hat{S}_E^B\left(X',X\right),\hat{S}^B\left(X',X\right),S_E^B\left(X,X\right),S^B\left(X,X\right),\bar{S}_E\left(X',X\right),\bar{S}^B\left(X',X\right)\right\}_{(X',X)\in G}\end{array}$$

where the collection of groups $\{G\}$ describe a certain organization of the sector space S.

The organization of the sector space into groups $\{G\}$ is not unique: there are an infinite number of such organizations. Moreover, for each organization, multiple distinct phases are possible for each group: the uncertainty combined with the interconnectivity of sectors, leads to an infinite number of possible solutions, among which, some correspond to default states[67]. Thus, there are an infinite number of possible collective states, and each collective state is one among an infinite

[67] See Gosselin and Lotz (2024).

number of possible configurations of the full system, as composed of multiple groups, each of them in a possible phase.

Any group associated with the phase it is in is called a *sub-collective state*. For any group, there may be several sub-collective states. The following results are to be considered for each group.

A1.4.3 Caracteristics of collective states

We present the results without providing all explicit formulas, referring the reader instead to the appendices or to Gosselin and Lotz (2025a,b). The purpose is to show that a sub-collective state consists of a collection of stakes associated with the elements of the group, which describe the basis variables of the collective state, whereas the other quantities—returns, capital, and so forth—are functions depending on this basis variables and characterizing the fiber variables. The elements of the fiber are fixed by the basis in the collective state under consideration, but they will become dynamical variables again when we study fluctuations of the collective states.

A1.4.3.1 Averages For a given sub-collective state, the averages depend on firms' productivity and on uncertainty. For a given group, the averages of the stakes are denoted:

$$\Big\{\Big\langle\hat{S}_E\left(X',X\right)\Big\rangle,\Big\langle\hat{S}\left(X',X\right)\Big\rangle,\left\langle S_E\left(X,X\right)\right\rangle,\left\langle S\left(X,X\right)\right\rangle,$$
$$\Big\langle\hat{S}_E^B\left(X',X\right)\Big\rangle,\Big\langle\hat{S}^B\left(X',X\right)\Big\rangle,\left\langle S_E^B\left(X,X\right)\right\rangle,\left\langle S^B\left(X,X\right)\right\rangle,\left\langle\bar{S}_E\left(X',X\right)\right\rangle,\left\langle\bar{S}^B\left(X',X\right)\right\rangle\Big\}_{(X',X)\in G}$$

A sub collective state, i.e. the distribution:

$$\cup_G\Big\{\hat{S}_E\left(X',X\right),\hat{S}\left(X',X\right),S_E\left(X,X\right),S\left(X,X\right),$$
$$\hat{S}_E^B\left(X',X\right),\hat{S}^B\left(X',X\right),S_E^B\left(X,X\right),S^B\left(X,X\right),\bar{S}_E\left(X',X\right),\bar{S}^B\left(X',X\right)\Big\}_{(X',X)\in G}$$

is organized around these averages.

On average, the levels of shares and loans depend on uncertainty and on the firms' average returns within the group. These averages make it possible to determine the average returns and the average levels of available capital $\langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^2$ and $\Big\langle\hat{K}\Big\rangle\left\|\hat{\Psi}\right\|^2$ for the group, as given in Appendix 3. Once these averages are known, one can determine the levels of returns and stakes by sector, that is, for any sub-collective state.

A1.4.3.2 Investors and banks' returns and stakes Once the averages of stakes and returns of investors and banks have been obtained, the values of returns and stakes can be derived. They are obtained by solving the return equations and, once the sectoral returns are determined, by substituting them into (107) and (109) . The return equations generally yield two possible solutions per sector and per agent type:

$$\hat{R}_{exc}^L\left(X\right),\hat{R}_{exc}^H\left(X\right)$$
$$\bar{R}_{exc}^L\left(X\right),\bar{R}_{exc}^H\left(X\right)$$

This corresponds, for a given average, to a multiplicity of distribution of stakes, high or low:

$$\cup_\alpha\quad\Big\{\hat{S}_E^{H/L}\left(X',X\right),\hat{S}^{H/L}\left(X',X\right),S_E^{H/L}\left(X,X\right),S^{H/L}\left(X,X\right),\tag{115}$$
$$\hat{S}_E^{BH/L}\left(X',X\right),\hat{S}^{BH/L}\left(X',X\right),S_E^{BH/L}\left(X,X\right),S^BH/L\left(X,X\right),\bar{S}_E^{H/L}\left(X',X\right),\bar{S}^{BH/L}\left(X',X\right)\Big\}_{(X',X)\in G_\alpha}$$

for a given group.

Given that there are multiple types of possible returns per sector, there exists an infinite number of collective states possible for a group, on average. The values for the relevant variables are provided in Appendix 3.

Each collective state is specified by a possible collection (115) and such collection forms the basis of the collective state; returns, disposable capital per sector, and other such quantities, are functions of these data and constitute the fiber of the collective state.

A1.4.4 Background fields and disposable capital for investors and banks

A1.4.4.1 Background fields The backgrounds fields $\bar{\Psi}$ and $\hat{\Psi}$ are found in terms of returns in Gosselin and Lotz (2024). The solutions of (99) and (100) for the background fields are:

$$\begin{aligned}\left|\hat{\Psi}\left(\hat{K}_1,X_1\right)\right|^2 &= \left\|\hat{\Psi}_0\left(X_1\right)\right\|^2-\hat{\mu}\left\{\left(\frac{\hat{K}_1^2\hat{g}^2\left(X_1\right)}{2\sigma_{\hat{K}}^2}+\frac{\hat{g}\left(X_1\right)}{2}\right)\right.\\ &\quad\left.+\left(\frac{\left\langle\bar{K}\right\rangle^2\left\langle\bar{g}\right\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\Delta\left(\hat{k}^B\left(X_1,\left\langle X\right\rangle\right)A\right)\frac{\left\|\bar{\Psi}\right\|^2\hat{K}_1}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}-\left(\frac{\left\langle\hat{K}\right\rangle^2\left\langle\hat{g}\right\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left\langle\hat{g}^{ef}\right\rangle\frac{\hat{K}_1}{\left\langle\hat{K}\right\rangle}\right\}\end{aligned}$$

$$\begin{aligned}\left|\bar{\Psi}\left(\bar{K}_1,X_1\right)\right|^2 &= \left|\bar{\Psi}_0\left(X_1\right)\right|^2-\hat{\mu}\left\{\left(\frac{\bar{K}_1^2\bar{g}^2\left(X_1\right)}{\sigma_{\hat{K}}^2}+\frac{\bar{g}\left(X_1\right)}{2}\right)\right.\\ &\quad+\left(\frac{\left\langle\bar{K}\right\rangle^2\left\langle\bar{g}\right\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left(\left\langle\underline{\hat{k}}^B\right\rangle^{ef}\Delta\left(\hat{k}^B\left(X_1,\left\langle X\right\rangle\right)A\right)+\frac{\Delta\bar{k}_2\left(\left\langle X\right\rangle,X\right)}{\left(1-\left\langle\bar{k}_1\right\rangle\right)\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}\left\langle\bar{g}\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^2\bar{K}_1}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\\ &\quad\left.-\left(\frac{\left\langle\hat{K}\right\rangle^2\left\langle\hat{g}\right\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left\langle\hat{g}^{Bef}\right\rangle\frac{\bar{K}_1}{\left\langle\hat{K}\right\rangle}\right\}\end{aligned}$$

where, for any quantity Q, the notation $\Delta\left(Q\left(X\right)\right)$ stands for its deviation from the average:

$$\Delta\left(Q\left(X\right)\right)=\left(Q\left(X\right)\right)-\left\langle\Delta\left(Q\left(X\right)\right)\right\rangle$$

The background field for firms is obtained similarly as solution (101):

$$\left|\Psi\left(K,X\right)\right|^2=\left|\Psi_0\left(X\right)\right|^2-\epsilon\left(\frac{\left(f_1^{(e)}\left(X\right)K-\bar{C}\left(X\right)\right)^2}{\sigma_{\hat{K}}^2}+\frac{f_1^{(e)}\left(X\right)}{2}\right) \tag{116}$$

More importantly for the description of the collective state are the average over capital of the fields by sector. These are obtained for given levels of stakes characteristic of a collective state. We obtain the following formula for the average sectoral fields:

$$\left|\hat{\Psi}_0\left(X\right)\right|^2\simeq\frac{\left\|\hat{\Psi}\right\|^2\left(I_{X/\left\langle X'\right\rangle}\right)^{\frac{3}{2}}}{\left(1-\bar{S}\right)^{\frac{3}{2}}\left(\frac{\hat{f}(X)}{\left\langle\hat{f}\right\rangle}+\frac{\left\langle\hat{S}(X',X)\right\rangle_{X'}}{1-\left\langle\hat{S}\right\rangle}+\frac{\left(\left\langle\hat{S}_1^B(X,X')\right\rangle_{X'}+\left\langle\hat{S}_2^B(X,X')\right\rangle_{X'}+\frac{\left\langle\hat{S}_1^B\right\rangle+\left\langle\hat{S}_2^B\right\rangle}{1-\left\langle\hat{S}\right\rangle}\right)\frac{\left\langle\bar{K}\right\rangle\left\|\bar{\Psi}\right\|^2}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\left\langle\bar{f}\right\rangle}{\left(1-\left\langle\bar{S}\right\rangle\right)\left\langle\bar{f}\right\rangle}\right)^{\frac{3}{2}}} \tag{117}$$

$$\left|\bar{\Psi}_0(X)\right|^2 \simeq \left(\frac{\langle\bar{f}\rangle \frac{\langle\bar{S}(X',X)\rangle_X}{1-\bar{S}(X',X)}}{\left(\left(1-\langle\bar{S}\rangle\right)\bar{f}(X)+\langle\bar{S}(X',X)\rangle_{X'}\langle\bar{f}\rangle\right)\frac{\langle\bar{S}(X',X)\rangle}{1-\langle\bar{S}(X',X)\rangle}}\right)^{\frac{3}{2}} \left\|\bar{\Psi}\right\|^2 \tag{118}$$

where:

$$I_{X/\langle X'\rangle} = \frac{\frac{\langle\hat{S}(X,X')\rangle_{X'}}{1-\langle\hat{S}(X,X')\rangle_{X'}}}{\frac{\langle\hat{S}(X,X')\rangle}{1-\langle\hat{S}(X,X')\rangle}} \tag{119}$$

These solutions depends on the stakes and their average[68] as well as the average fields[69] $\left\|\hat{\Psi}\right\|^2$ and $\left\|\bar{\Psi}\right\|^2$.

The results depend on the stakes and their average over the sector-space and are replaced in the return equations.

A1.4.4.2 Disposable capital For a given distribution of stakes and returns, and using the average disposable capital $\langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^2$ and $\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2$ defined in Gosselin and Lotz (2025b), the disposable capital per sector writes

$$\bar{K}_X\left|\bar{\Psi}(X)\right|^2 \simeq \langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^2\left(\frac{\langle\bar{f}\rangle \frac{\langle\bar{S}(X',X)\rangle_X}{1-\bar{S}(X',X)}}{\left(\left(1-\langle\bar{S}\rangle\right)\bar{f}(X)+\langle\bar{S}(X',X)\rangle_{X'}\langle\bar{f}\rangle\right)\frac{\langle\bar{S}(X',X)\rangle}{1-\langle\bar{S}(X',X)\rangle}}\right)^2 \tag{120}$$

$$\hat{K}_X\left|\hat{\Psi}(X)\right|^2 \simeq \frac{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2\left(\frac{\langle\hat{S}(X',X)\rangle_X}{1-\langle\hat{S}(X',X)\rangle_X}\frac{1-\langle\hat{S}(X',X)\rangle}{\langle\hat{S}(X',X)\rangle}\langle\bar{f}\rangle\right)^2}{\left(1-\bar{S}\right)^2\left(\hat{f}(X)+\frac{\langle\hat{S}(X',X)\rangle_{X'}\langle\hat{f}\rangle}{1-\langle\hat{S}\rangle}+\frac{\left(\langle\hat{S}_1^B(X,X')\rangle_{X'}+\langle\hat{S}_2^B(X,X')\rangle_{X'}+\frac{\langle\hat{S}_1^B\rangle+\langle\hat{S}_2^B\rangle}{1-\langle\hat{S}\rangle}\right)\frac{\langle\bar{K}\rangle\|\bar{\Psi}\|^2}{\langle\hat{K}\rangle\|\hat{\Psi}\|^2}\langle\bar{f}\rangle}{1-\langle\bar{S}\rangle}\right)^2} \tag{121}$$

Here too, results depend on the value of stakes, justifying the denomination of fiber states.

A1.5 Resolution and collective states: default scenario

Defaults states can be defined as modifications of non-default states. Assuming one or several defaulting firms initially in one sector, the default state is built recursively from this initial impact. Once the initial default has propagated, the returns in each sector will be deviations from what would have been the returns in a non-default scenario.

We will detail below the propagation mechanism and the condition for the default to materialize and spread. The loss incured by investors and the fraction of defaulting investors will then be computed.

A1.5.1 Principle of propagation

The default states for investors have been described in Gosselin and Lotz (2025). The principle of propagation is similar if we include banks. The default states are found recursively. We first

[68] See Gosselin and Lotz (2025a,b)

[69] Computed in Gosselin and Lotz (2025a,b).

assume some realized initial default states S_0, $\hat{S}_0$ and $\bar{S}_0$, when a set of firms, investors and banks experience a complete loss of their private capital. In terms of returns this corresponds to:

$$\begin{aligned}
\hat{f}(X') &= -1 \\
\bar{f}(X') &= -1 \\
f_1'(X) &= -1
\end{aligned} \tag{122}$$

for investors, banks and firms in $\hat{S}_0$, $\bar{S}_0$ and S_0 respectively. The number -1 stands for a return of -100 percent, so that the private capital for defaulting firms, banks and investors are 0.

The possible default sets are then obtained from this initial set by the limit of a sequence of equations. We assume that after n iterations, the sets of defaulting agents are S_n, $\hat{S}_n$ and $\bar{S}_n$. We define the remaining sets of agents as $S/\hat{S}_n$, $S/\bar{S}_n$ and S/S_n.

Then these sets are included in the return equations for investors and banks:

A1.5.1.1 Investors

$$\begin{aligned}
0 &= \int \left(\delta(X-X') - \hat{S}_E(X',X)\right)\widehat{DF}(X')\left(\hat{f}_{n+1}(X') - \bar{r}\right)dX' \\
&+ \int_{\hat{S}_n} \frac{1-\left(\hat{S}(X') + \hat{S}_E^B(X') + \hat{S}_L^B(X')\right)}{\hat{S}_L(X')}\hat{S}_L(X',X)\,dX' \\
&+ \int_{S_n} \frac{1-\left(S(X') + S_E^B(X') + S_L^B(X')\right)}{S_L(X')}S_L(X',X)\,dX' - \int S_E(X',X)\left((f(X') - \bar{r})\right)dX'
\end{aligned} \tag{123}$$

A1.5.1.2 Banks

$$\begin{aligned}
0 &= \left(\delta(X-X') - \bar{S}_E(X',X)\right)\left(\bar{f}_{n+1}(X') - \bar{r}\right)\overline{DF}(X') - \widehat{DF}(X')\hat{S}_E^B(X',X)\left(\hat{f}(X') - \bar{r}\right) \\
&+ \int_{\bar{S}_n} \bar{S}_L(X',X)\frac{\left(1-\bar{S}(X')\right)}{\bar{S}_L(X')} + \int_{\hat{S}_n} \hat{S}_L^B(X',X)\frac{1-\left(S(X') + S_E^B(X') + S_L^B(X')\right)}{S_L(X') + S_L^B(X')} \\
&+ \int_{S_n} S_L^B(X',X)\frac{1-\left(\hat{S}(X',X) + \hat{S}_E^B(X') + \hat{S}_L^B(X')\right)}{\hat{S}_L(X') + \hat{S}_L^B(X')} - S_E^B(X',X)\left(f(X') - \bar{r}\right)
\end{aligned} \tag{124}$$

These equations define new sets of returns $\hat{f}_{n+1}(X')$ and $\bar{f}_{n+1}(X')$ which in turn may define new default sets S_{n+1}, $\hat{S}_{n+1}$ and $\bar{S}_{n+1}$. Once the iteration stabilizes, we find a remaining sets of - non-defaulted - agents $\left(\hat{S}_\infty, \bar{S}_\infty, S_\infty\right)$, defined by the limit $\left(S/\hat{S}_n, \bar{S}/\bar{S}_n, S/S_n\right) \underset{n\to\infty}{\rightarrow} \left(\hat{S}_\infty, \bar{S}_\infty, S_\infty\right)$ with returns $\left\{\hat{f}_n(X) \to \hat{f}(X), \bar{f}_n(X) \to \bar{f}(X)\right\}$ for which the resulting disposable capitals are given by (190), (189) and (192).

A1.5.2 Conditions for propagation

Firms default when they meet the conditions for initial defaults. But, for this default to propagate to the whole set of investors and banks, some additional conditions must be met. First, the defaut must propagate to the firm's immediate investors or banks. Then, it must propagate from these investors to other investors and banks. The case of investors default was considered in Gosselin and Lotz (2025a) and we focus on the banks default only[70].

[70] See Appendix 11.3.

A1.5.2.1 Propagation from firms to banks and investors Some firms may lack the disposable capital to face their costs, pushing them into default. Whether these initial defaults may push into default their intra-sectoral banks depends on the magnitude of loss of the defaulting firms.

For the default to spread to the firm's banks, the level of loss a firm has to experience, by lack of capital or increase in costs, must be below a negative threshold[71]:

$$R(X) < D_{Th}$$

where the threshold D_{Th} is:

$$\begin{aligned} D_{Th} &= -\frac{1}{S_E^B(X,X)}\left(1+\bar{r}+\left\langle \bar{S}_E(X',X)\right\rangle_{X'}\overline{DF}(X')\left(\left\langle \bar{f}(X')\right\rangle-\bar{r}\right)\right. \\ &\quad \left.+\left\langle \hat{S}_E^B(X',X)\right\rangle_{X'}\widehat{DF}(X')\left(\left\langle \hat{f}(X')\right\rangle-\bar{r}\right)\right) \end{aligned}$$

This threshold is negative, and the default zone includes all returns that are below it.

A1.5.2.2 Propagation from investors and banks to other sectors Given an initial default in investors and banks in certain sectors, the condition for default to propagate to investors is the same as in part 1 and the condition under which the default may spread to other banks is:

$$\begin{aligned} &\mathbf{H}\left(\left\langle\left(\hat{S}_L(X',X)\right)\right\rangle_{X'}+S_L(X,X)\right) \\ &+\mathbf{G}\left(\left\langle \bar{S}_L(X',X)\right\rangle_{X'}+\left\langle \hat{S}_L^B(X',X)\right\rangle_{X'}+S_L^B(X,X)\right)>2\left\langle \bar{f}\right\rangle \end{aligned} \tag{125}$$

where $\mathbf{H}$ and $\mathbf{G}$ are functions of stakes, uncertainty and capital ratios under the non-default scenario. They are given by:

$$\mathbf{H}=\frac{1-S_1^B(X,X)\frac{A_1}{B}dC}{1-\bar{S}(X')}+H\left(1-S_1(X,X)\frac{A_1}{B}dC\right)$$

$$\mathbf{G}=\left(\frac{1-S_1^B(X,X)\frac{A_2}{B}dC}{1-\bar{S}(X')}\right)$$

and:

$$dC=\left(C-\frac{r\left(f_1(X)+\tau\Delta F_\tau\left(\bar{R}(K,X)\right)\right)}{(1-S(X))^{1-r}}\right)$$

The term:

$$H=\frac{S_1^B(X,X)}{1-\bar{S}(X')}\frac{\hat{S}_1^B(X',X)}{1-\left\langle \hat{S}(X')\right\rangle}\frac{1-\left(\hat{S}(X')+\hat{S}_1^B(X')+\hat{S}_2^B(X')\right)}{1-\left(\hat{S}_1(X')+\hat{S}_1^B(X')\right)}$$

increases with participations $\hat{S}_1(X')+\hat{S}_1^B(X')$ and $\bar{S}(X')$ and $\left\langle \hat{S}(X')\right\rangle$, while it decreases with loans $\hat{S}_2(X')+\hat{S}_2^B(X')$.

The coefficients A_1 and A_2:

$$A_1=\frac{\frac{h(X)\hat{h}(X)}{2}\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}+h_1^B(X)\left\langle \hat{h}_1^B(X)\right\rangle\frac{\bar{K}_X|\bar{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}{1-\left\langle \hat{S}(X)\right\rangle},\ A_2=h_1^B(X)\left\langle \bar{h}(X)\right\rangle_{\bar{S}}\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}$$

[71] See Appendix 8 in Gosselin and Lotz (2025b).

increase with $\left\langle \hat{S}(X)\right\rangle$, with capital ratios $\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}$ $\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}$ and, decrease with uncertainty since $h_1^B(X)\left\langle \hat{h}_1^B(X)\right\rangle$ and $h_1^B(X)\left\langle \bar{h}(X)\right\rangle_{\bar{S}}$ decrease with uncertainty γ.

$$\begin{aligned}
B &= \left(1-2\frac{S(X)+S^B(X)}{1-\left(S(X)+S^B(X)\right)}\right)\\
&\times\left(1-dC\frac{\left(\frac{h(X)\hat{h}(X)}{2}\left(1-\frac{\langle S_1(X,X)\rangle}{1-\left\langle \hat{S}(X)\right\rangle}\right)\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}+h_1^B(X)\left\langle \bar{h}(X)\right\rangle_{f(X)}\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)}{1-2\frac{S(X)+S^B(X)}{1-\left(S(X)+S^B(X)\right)}}\right)
\end{aligned}$$

B increases with uncertainty γ. Then using that

$$\frac{\langle S_1(X,X)\rangle}{1-\left\langle \hat{S}(X)\right\rangle}=\frac{1-\left\langle \hat{S}(X)\right\rangle-\langle S_2(X,X)\rangle}{1-\left\langle \hat{S}(X)\right\rangle}=1-\frac{\langle S_2(X,X)\rangle}{1-\left\langle \hat{S}(X)\right\rangle}$$

decreases with $\left\langle \hat{S}(X)\right\rangle$, we obtain that B decreases with $\left\langle \hat{S}(X)\right\rangle$. B also decreases with capital ratios $\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}$ $\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}$.

As a consequence, **G** increases as function of $\bar{S}(X')$, $\left\langle \hat{S}(X)\right\rangle$, with capital ratios $\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}$ $\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}$ and decrease as a function of $S_1^B(X,X)$ and uncertaint γ.

Moreover, **H** increases as function of $\bar{S}(X')$, $\left\langle \hat{S}(X)\right\rangle$, $\hat{S}_1(X')$, $\hat{S}_1^B(X')$, with capital ratios $\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}$, $\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}$ while it decreases with loans $\hat{S}_2(X')+\hat{S}_2^B(X')$ and decrease as a function of $S_1^B(X,X)$, $S_1(X,X)$ and uncertainty γ.

A1.5.3 Description of the default state

Once the default has materialized, the system is in a default state. This state can be analyzed as a deviation from a no-default state. To do so, we will compute, for the default state, both the average loss and the fraction of investors affected.

When defaults occur, investors' returns are shifted by:

$$\hat{f}(X)\rightarrow\hat{f}(X)-\mu d\hat{f} \tag{126}$$

where $d\hat{f}$ is the loss incured by the remaining investors for each investor defaulting, and μ is the fraction of investors impacted by the default.

We find that:

$$d\hat{f}(X)=\frac{\left(\left\langle \hat{S}_L(X',X)\right\rangle+\langle S_L(X,X)\rangle\right)\mu+\langle S_E(X')\rangle\langle df(X')\rangle}{1-\left\langle \hat{S}(X')\right\rangle}$$

and that the fraction of defaulting investors μ is:

$$\mu=\frac{1-\left\langle \hat{S}(X')\right\rangle}{\left(\left\langle \hat{S}_L(X',X)\right\rangle+\langle S_L(X,X)\rangle\right)+\langle S_E(X')\rangle\langle df(X')\rangle-2\left\langle \hat{f}\right\rangle\left(1-\left\langle \hat{S}(X')\right\rangle\right)} \tag{127}$$

where all parameter values are those of a non-default scenario.

When defaults occur, banks' returns are shifted by:

$$\bar{f}(X) \rightarrow \bar{f}(X) - \mu d\bar{f} \tag{128}$$

where $d\bar{f}$ is the loss incured by the remaining investors for each investor defaulting, and μ is the fraction of investors impacted by the default.

We find that:

$$\begin{aligned}
\frac{\left\langle d\bar{f}\right\rangle}{\mu} &= -\left(\frac{S_E^B(X,X)}{1-\bar{S}(X')} + S_E(X,X)\frac{S_E^B(X,X)}{1-\bar{S}(X')}\frac{\hat{S}_E^B(X',X)}{1-\left\langle\hat{S}(X')\right\rangle}\widehat{DF}(X')\right)\frac{A}{B}dC \\
&+\left(\frac{1}{1-\bar{S}(X')} + \frac{S_E^B(X,X)}{1-\bar{S}(X')}\frac{\hat{S}_E^B(X',X)}{1-\left\langle\hat{S}(X')\right\rangle}\widehat{DF}(X')\right)\left(\left\langle\left(\hat{S}_L(X',X)\right)\right\rangle_{X'} + S_L(X,X)\right) \\
&+\frac{1}{1-\bar{S}(X')}\left(\left\langle\bar{S}_L(X',X)\right\rangle_{X'} + \left\langle\hat{S}_L^B(X',X)\right\rangle_{X'} + S_L^B(X,X)\right)
\end{aligned}$$

and the fraction of defaulting investors μ is given by:

$$\mu = \frac{1}{\frac{\left\langle d\bar{f}\right\rangle}{\mu} - 2\left\langle\bar{f}\right\rangle} \tag{129}$$

where all parameter values are those of a non-default scenario.

$$\left\langle d\bar{f}\right\rangle = \frac{1}{\frac{\left\langle d\bar{f}\right\rangle}{\mu} - 2\left\langle\bar{f}\right\rangle}\frac{\left\langle d\bar{f}\right\rangle}{\mu}$$

The average loss due to default for the entire group of firms can also be derived:

$$\langle dR(X)\rangle = -\frac{A}{B}\mu dC \tag{130}$$

with:

$$A = A_1\left(\left\langle\left(\hat{S}_L(X',X)\right)\right\rangle_{X'} + S_L(X,X)\right) + A_2\left(\left\langle\bar{S}_L(X',X)\right\rangle_{X'} + \left\langle\hat{S}_L^B(X',X)\right\rangle_{X'} + S_L^B(X,X)\right)$$

$$A_1 = \frac{\frac{w(X)\hat{w}(X)}{2}\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2} + w_E^B(X)\left\langle\hat{w}_E^B(X)\right\rangle\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}{1-\left\langle\hat{S}(X)\right\rangle},\ A_2 = w_E^B(X)\langle\bar{w}(X)\rangle_{\bar{S}}\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}$$

$$\begin{aligned}
B &= \left(1-2\frac{S(X)+S^B(X)}{1-(S(X)+S^B(X))}\right) \\
&\times\left(1-dC\frac{\left(\frac{w(X)\hat{w}(X)}{2}\left(1-\frac{\langle S_E(X,X)\rangle}{1-\left\langle\hat{S}(X)\right\rangle}\right)\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2} + w_E^B(X)\langle\bar{w}(X)\rangle_{f(X)}\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)}{1-2\frac{S(X)+S^B(X)}{1-\left(S(X)+S^B(X)\right)}}\right)
\end{aligned}$$

The average loss that results from the default for the entire group of firms can also be derived:

$$\langle df(X')\rangle = -\frac{\left(\left\langle\hat{S}_L(X',X)\right\rangle + \langle S_L(X,X)\rangle\right)\frac{N}{1-\left\langle\hat{S}(X)\right\rangle}dC}{1-N\left(1-\frac{\langle S_E(X,X)\rangle}{1-\left\langle\hat{S}(X)\right\rangle}\right)dC}$$

where the parameter dC writes:

$$dC = C - \frac{r f_1(X)}{(K_X)^{r-1}} \tag{131}$$

This parameter reflects the impact of defaults on the firms' cost structure. It captures the relative weight of costs against marginal returns: higher values of dC are associated with reduced net capital and a higher likelihood of default.

Similarly, the expressions above reveal that, the higher the investment stake $S(X)$ in sector X, the smaller the value of dC. Financially vulnerable states—characterized by a high dC—are more prone to initiate a cascade of defaults. The resulting fraction of defaulting investors will be given by equation.

A1.6 Internal dynamics

To reveal the internal dynamics of the system, we transform the static return equations (102) and (103) for each group into a dynamic system. To do so, we perform a first-order perturbation of these return equations.

A1.6.1 Dynamic return equation

A1.6.1.1 Investors To study the distribution of stakes within groups dynamically, we must specify the temporal sequence of investments by replacing the static return equations (102), (103) with a dynamic formulation. For investors, this can be expressed as:

$$\begin{aligned} 0 &= \widehat{DF}(X',\theta-1)\left(\hat{f}(X',\theta)-\bar{r}\right) \\ &\quad -\left\langle \hat{S}_E(X',X,\theta)\right\rangle_{X'}\left\langle \widehat{DF}(X,\theta-1)\right\rangle\left(\left\langle \hat{f}(X',\theta)\right\rangle - \langle \hat{r}(X')\rangle\right) \\ &\quad -S_E(X,X,\theta-1)\left(f(X,\theta-1)-\langle \bar{r}(X)\rangle\right) \end{aligned} \tag{132}$$

where $\widehat{DF}(X,\theta-1)$ denotes the dynamic discount factor of investor X for debt repayment:

$$\widehat{DF}(X,\theta-1) = \frac{1-\left\langle \hat{S}(X',\theta-1)\right\rangle + \left\langle \hat{S}_E^B(X',\theta-1)\right\rangle + \left\langle \hat{S}_L^B(X',\theta-1)\right\rangle}{1-\left\langle \hat{S}_E(X',\theta-1)\right\rangle + \left\langle \hat{S}_E^B(X',\theta-1)\right\rangle}$$

and the firms' returns are assumed to exhibit decreasing returns to scale:

$$f(X,\theta-1) - \langle \bar{r}(X)\rangle = \frac{f_1(X,\theta-1)+\tau\left(\langle f_1(X)\rangle - \langle f_1(X')\rangle\right)}{\left(K(X,,\theta-1)\left\|\Psi((X,,\theta-1))\right\|^2\right)^r} - \frac{C}{K(X,,\theta-1)} - C_0 - \langle \bar{r}(X)\rangle$$

A1.6.1.2 Banks The dynamic equation for the bank's return is given by:

$$\begin{aligned} 0 &= \overline{DF}(X,\theta-1)\left(\bar{f}(X,\theta)-(1+\kappa)\bar{r}\right) \\ &\quad -\left\langle \bar{S}_E(X',X,\theta-1)\right\rangle_{X'}\left\langle \overline{DF}(X',\theta-1)\right\rangle\left(\left\langle \bar{f}(X',\theta)\right\rangle-(1+\kappa)\bar{r}\right) \\ &\quad -\int\left\langle \hat{S}_E^B(X',X,\theta-1)\right\rangle_{X'}\left\langle \widehat{DF}(X',\theta-1)\right\rangle\left(\left\langle \hat{f}(X',\theta)\right\rangle-\langle \hat{r}(X')\rangle\right)dX' \\ &\quad -S_E^B(X,X,\theta-1)\left(f(X,\theta-1)-\langle \bar{r}(X)\rangle\right) \end{aligned} \tag{133}$$

where $\overline{DF}(X,\theta-1)$ denotes the dynamic discount factor of investor X for debt repayment:

$$\overline{DF}(X,\theta-1) = \frac{1-\bar{S}(X,\theta-1)}{1-\bar{S}_E(X,\theta-1)}$$

and the returns they generate:

$$\hat{f}\left(X^{\prime},\theta\right)-\bar{r}$$

The returns generated by firms are themselves determined by lagged productivity:

$$f\left(X,\theta-1\right)-\bar{r}$$

so that capital reallocation delays the effect on firms' returns.[72][73].

13 Appendix 2 Some formulas for field models

13.1 A2.1 Field models for investors and banks

A2.1.1 Micro model for investors and banks

The micro-foundations of investors and banks dynamics have been developed in Gosselin and Lotz (2024). We present the main elements leading to the field model for $\hat{\Psi}$ and $\bar{\Psi}$.

The dynamics for $\hat{K}_j\left(t\right)$ is described by the equation:

$$\frac{d}{dt}\hat{K}_j\left(t\right)=\hat{R}_j+\sum_l\hat{M}_{jl}\frac{d}{dt}\hat{K}_l\left(t\right)-\sum_l\bar{N}_{jl}\frac{d}{dt}\bar{K}_{0l}\left(t\right)$$

The right-hand side of equation (134) indicates that the variation in investor j's disposable capital depends not only on its return $\hat{R}_j$, but also on the capital provided by other investors through the matrix $\hat{M}$, as well as that provided by banks through the matrix $\bar{N}$. This equation can be reformulated as a dynamic process involving the variation of all the capital at the investor's disposal:

$$\sum_l\left(\delta_{jl}-\hat{M}\right)_{jl}\frac{d}{dt}\hat{K}_l\left(t\right)+\sum_l\bar{N}_{jl}\frac{d}{dt}\bar{K}_{0l}\left(t\right)=\hat{R}_j \tag{134}$$

where:

$$\hat{M}_{jm}=\frac{\hat{k}_{jm}\hat{K}_j\left(t\right)}{1+\sum_l\hat{k}_{jl}\hat{K}_l\left(t\right)+\sum_l\hat{k}_{1jl}^B\bar{K}_{l0}\left(t\right)+\kappa\sum_l\hat{k}_{2jl}^B\frac{\bar{K}_{l0}(t)}{1+\sum_m\bar{\bar{k}}_{lm}\bar{K}_{m0}(t)}} \tag{135}$$

and:

$$\bar{N}_{jl}=\frac{\left(\hat{k}_{1jl}^B+\kappa\hat{k}_{2jl}^B\frac{1}{1+\sum_m\bar{\bar{k}}_{lm}\bar{K}_{m0}(t)}-\kappa\frac{\sum_m\hat{k}_{2jm}^B\bar{K}_{m0}(t)\bar{\bar{k}}_{ml}}{\left(1+\sum_n\bar{\bar{k}}_{mn}\bar{K}_{n0}(t)\right)^2}\right)\hat{K}_j}{1+\sum_l\hat{k}_{jl}\hat{K}_l\left(t\right)+\sum_l\hat{k}_{1jl}^B\bar{K}_{l0}\left(t\right)+\kappa\sum_l\hat{k}_{2jl}^B\frac{\bar{K}_{j0}(t)}{1+\sum_m\bar{\bar{k}}_{lm}\bar{K}_{m0}(t)}} \tag{136}$$

In the continuous approximation, the capital accumulation for banks writes:

$$\frac{d}{dt}\bar{K}_{0j}\left(t\right)=\bar{R}_j+\sum_l M_{jl}\frac{d}{dt}\bar{K}_{0l}\left(t\right)$$

Similarly to investors, the variation in available capital for bank j is the sum of its total return, denoted as $\bar{R}_j$, plus the variation in participations and loans from banks investing in bank j, through the matrix $\bar{M}$. As before, this equation can be reformulated as a dynamic process involving the variation of all the capital at bank j's disposal.

$$\sum_l\left(\delta_{jl}-M_{jl}\frac{d}{dt}\right)\bar{K}_{0l}\left(t\right)=\bar{R}_j^{\prime} \tag{137}$$

[72] In this study, we focus on the case of decreasing returns to scale.

[73] The formula for $f\left(\hat{X},\theta-1\right)-\bar{r}$ is provided in Appendix 9 in Gosselin and Lotz (2025).

with:

$$\bar{M}_{jm} = \frac{\bar{k}_{jm}\bar{K}_{j0}(t)}{1+\sum_{\nu}\bar{k}_{jv}\hat{K}_{\nu}(t)} \tag{138}$$

Here, the overall leverage effect $\bar{k}_{jm}$ provided by bank m to bank j decomposes into the leverage effect associated with participations and loans, denoted as $\bar{k}_{1jm}$ and $\bar{k}_{2jm}$ respectively:

$$\bar{k}_{1jm} + \bar{k}_{2jm} = \bar{k}_{jm}$$

A.2.1.2 Field translation

A.2.1.2.1 Field translation in term of exogenous connections We showed previously that the dynamic equations (134), (137) can be translated into field action functional encompassing the dynamical accumulation for each sector. Consider first the field $\bar{\Psi}$ describing the banking system. It depends on the two variables $\bar{K}$ and X, and its action functional is given by:

$$-\bar{\Psi}^{\dagger}\left(\bar{K},X\right)\nabla^{2}\bar{\Psi}\left(\bar{K},X\right)+\left(\frac{\bar{g}^{2}\left(\bar{K},X\right)}{2\sigma_{\bar{K}}^{2}}+\frac{\bar{g}\left(\bar{K},X\right)}{2\bar{K}}\right)\left|\bar{\Psi}\left(\bar{K},X\right)\right|^{2} \tag{139}$$

where:

$$\bar{g}\left(\hat{K},X\right)=\left(1-\bar{M}\left|\bar{\Psi}\left(\bar{K},X\right)\right|^{2}\right)^{-1}\bar{f}\left(\bar{K},X\right)$$

and $\bar{M}$ is the matrix translating its microeconomic equivalent (135).

The field $\hat{\Psi}$ describing the investors depends on the two variables $\hat{K}$ and X, and its action functional is given by:

$$-\hat{\Psi}^{\dagger}\left(\hat{K},X\right)\nabla^{2}\hat{\Psi}\left(\hat{K},X\right)+\left(\frac{\hat{g}^{2}\left(\hat{K},X\right)}{2\sigma_{\hat{K}}^{2}}+\frac{\hat{g}\left(\hat{K},X\right)}{2\hat{K}}\right)\left|\hat{\Psi}\left(\hat{K},X\right)\right|^{2} \tag{140}$$

where we define:

$$\hat{g}\left(\hat{K},X\right)=\left(1-\hat{M}\left|\hat{\Psi}\left(\hat{K},X\right)\right|^{2}\right)^{-1}\hat{f}\left(\hat{K},X\right)+\left(1-\hat{M}\right)^{-1}\bar{N}\left(1-\bar{M}\right)^{-1}\bar{f}\left(\bar{K},X\right)$$

The field definition of the matrices $\bar{M}$, $\hat{M}$ and $\bar{N}$ are given here.

These matrices $\bar{M}\left(\left(\bar{K},X\right),\left(\bar{K}',X'\right)\right)$, $\hat{M}\left(\left(\hat{K}',X'\right),\left(\hat{K},X\right)\right)$ and $\bar{N}\left(\left(\hat{K},X\right),\left(\bar{K},X\right)\right)$, were defined in GL 24 using exogenous set of connections. They were the direct translation of (135), (136), (138):

$$\bar{M}\left(\left(\bar{K},X\right),\left(\bar{K}',X'\right)\right)=\frac{\bar{\bar{k}}\left(X,X'\right)\bar{K}_{0}}{1+\int\bar{\bar{k}}\left(X,X'\right)\bar{K}_{0}'\left|\bar{\Psi}\left(\bar{K}_{0}',X'\right)\right|^{2}}$$

$$\begin{aligned}&\hat{M}\left(\left(\hat{K}',X'\right),\left(\hat{K},X\right)\right)\\=&\frac{\hat{k}\left(X,X'\right)\hat{K}}{1+\int\hat{k}\left(X,X'\right)\left|\hat{\Psi}\left(\hat{K}',X'\right)\right|^{2}+\int\hat{k}_{1}^{B}\left(X,X'\right)\bar{K}_{0}'\left|\bar{\Psi}\left(\bar{K}_{0}',X'\right)\right|^{2}+\kappa\int\hat{k}_{2}^{B}\left(X,X'\right)\frac{\bar{K}_{0}'\left|\bar{\Psi}\left(\bar{K}_{0}',X'\right)\right|^{2}}{1+\int\bar{\bar{k}}\left(X'\right)\bar{K}_{0}''\left|\bar{\Psi}\left(\bar{K}_{0}'',X''\right)\right|^{2}}}\end{aligned}$$

and ultimately:

$$\bar{N}=\frac{\left(\hat{k}_{1}^{B}\left(X,X'\right)+\kappa\frac{\hat{k}_{2}^{B}\left(X,X'\right)}{1+\int\bar{\bar{k}}\left(X'\right)\bar{K}_{0}''\left|\bar{\Psi}\left(\bar{K}_{0}'',X''\right)\right|^{2}}-\kappa\int\frac{\hat{k}_{2}^{B}\left(X,X''\right)\bar{K}_{0}''\bar{\bar{k}}\left(X'',X'\right)}{\left(1+\int\bar{\bar{k}}\left(X'',\bar{Y}\right)\bar{K}_{0}^{Y}\left|\bar{\Psi}\left(\bar{K}_{0}^{Y},\bar{Y}\right)\right|^{2}\right)^{2}}\right)\hat{K}}{1+\int\hat{k}\left(X,X'\right)\left|\hat{\Psi}\left(\hat{K}',X'\right)\right|^{2}+\int\hat{k}_{1}^{B}\left(X,X'\right)\bar{K}_{0}'\left|\bar{\Psi}\left(\bar{K}_{0}',X'\right)\right|^{2}+\kappa\int\hat{k}_{2}^{B}\left(X,X'\right)\frac{\bar{K}_{0}'\left|\bar{\Psi}\left(\bar{K}_{0}',X'\right)\right|^{2}}{1+\int\bar{\bar{k}}\left(X'\right)\bar{K}_{0}''\left|\bar{\Psi}\left(\bar{K}_{0}'',X''\right)\right|^{2}}}$$

A.2.1.2.1 Field translation in term of endogenous stakes Using the endogenous stakes defined in Gosselin and Lotz (2025b), we have:

the various matrices defined in the first part and arising in the return equation can be rewritten in terms of stakes (see Gosselin and Lotz (2025b)):

$$\hat{S}_{\eta}\left(X',\hat{K}',X\right) = \frac{\hat{K}'\hat{k}_{\eta}\left(X',X\right)\left|\hat{\Psi}\left(\hat{K}',X'\right)\right|^{2}}{1+\underline{\hat{k}}\left(X'\right)+\underline{\hat{k}}_{1}^{B}\left(X'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\underline{\hat{k}}}\right]\left(X'\right)} \tag{141}$$

$$\hat{S}_{\eta}\left(X',X\right) = \int\frac{\hat{K}'\hat{k}_{\eta}\left(X',X\right)\left|\hat{\Psi}\left(\hat{K}',X'\right)\right|^{2}}{1+\underline{\hat{k}}\left(X'\right)+\underline{\hat{k}}_{1}^{B}\left(X'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\underline{\hat{k}}}\right]\left(X'\right)}d\hat{K}' = \frac{\hat{K}_{X'}\hat{k}_{\eta}\left(X',X\right)\left|\hat{\Psi}\left(X'\right)\right|^{2}}{1+\underline{\hat{k}}\left(X'\right)+\underline{\hat{k}}_{1}^{B}\left(X'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\underline{\hat{k}}}\right]\left(X'\right)}$$

and:

$$S_{\eta}\left(X',K',X\right) = \frac{k_{\eta}\left(X',X\right)K'\left|\Psi\left(K',X'\right)\right|^{2}}{1+\underline{k}\left(X'\right)+\underline{k}_{1}^{(B)}\left(X'\right)+\kappa\left[\frac{\underline{k}_{2}^{B}}{1+\underline{k}}\right]\left(X'\right)} \tag{142}$$

$$S_{\eta}\left(X',X\right) = \int\frac{k_{\eta}\left(X',X\right)K'\left|\Psi\left(K',X'\right)\right|^{2}}{1+\underline{k}\left(X'\right)+\underline{k}_{1}^{(B)}\left(X'\right)+\kappa\left[\frac{\underline{k}_{2}^{B}}{1+\underline{k}}\right]\left(X'\right)}dK' = \frac{k_{\eta}\left(X',X\right)K_{X'}\left|\Psi\left(X'\right)\right|^{2}}{1+\underline{k}\left(X'\right)+\underline{k}_{1}^{(B)}\left(X'\right)+\kappa\left[\frac{\underline{k}_{2}^{B}}{1+\underline{k}}\right]\left(X'\right)}$$

for investors, and:

$$\bar{S}_{\eta}\left(X',\bar{K}',X\right) = \frac{\bar{K}'\bar{k}_{\eta}\left(X',X\right)\left|\bar{\Psi}\left(\bar{K}',X'\right)\right|^{2}}{1+\underline{\bar{k}}\left(X'\right)}$$

$$\bar{S}_{\eta}\left(X',X\right) = \int\frac{\bar{K}'\bar{k}_{\eta}\left(X',X\right)\left|\bar{\Psi}\left(\bar{K}',X'\right)\right|^{2}}{1+\underline{\bar{k}}\left(X'\right)}d\bar{K}' = \frac{\bar{K}_{X'}\bar{k}_{\eta}\left(X',X\right)\left|\bar{\Psi}\left(X'\right)\right|^{2}}{1+\underline{\bar{k}}\left(X'\right)}$$

$$\hat{S}_{1}^{B}\left(X',\hat{K}',X\right) = \frac{\hat{K}'\underline{\hat{k}}_{1}^{B}\left(X',X\right)\left|\hat{\Psi}\left(\hat{K}',X'\right)\right|^{2}}{1+\underline{\hat{k}}\left(X'\right)+\underline{\hat{k}}_{1}^{B}\left(X'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\underline{\hat{k}}}\right]\left(X'\right)}$$

$$\hat{S}_{1}^{B}\left(X',X\right) = \frac{\hat{K}_{X'}\underline{\hat{k}}_{1}^{B}\left(X',X\right)\left|\hat{\Psi}\left(X'\right)\right|^{2}}{1+\underline{\hat{k}}\left(X'\right)+\underline{\hat{k}}_{1}^{B}\left(X'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\underline{\hat{k}}}\right]\left(X'\right)}$$

$$\hat{S}_{2}^{B}\left(X',\hat{K}',X\right) = \frac{\hat{K}'\frac{\kappa\underline{\hat{k}}_{2}^{B}\left(X',X\right)}{1+\underline{\hat{k}}(X)}\left|\hat{\Psi}\left(\hat{K}',X'\right)\right|^{2}}{1+\underline{\hat{k}}\left(X'\right)+\underline{\hat{k}}_{1}^{B}\left(X'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\underline{\hat{k}}}\right]\left(X'\right)}$$

$$\hat{S}_{2}^{B}\left(X',X\right) = \frac{\hat{K}_{X'}\frac{\kappa\underline{\hat{k}}_{2}^{B}\left(X',X\right)}{1+\underline{\hat{k}}(X)}\left|\hat{\Psi}\left(X'\right)\right|^{2}}{1+\underline{\hat{k}}\left(X'\right)+\underline{\hat{k}}_{1}^{B}\left(X'\right)+\kappa\frac{\underline{\hat{k}}_{2}^{B}\left(X'\right)}{1+\underline{\hat{k}}}}$$

$$\left[\frac{\underline{k}_{2}^{B}}{1+\underline{\bar{k}}}\right]\left(X'\right) = \int\frac{\underline{k}_{2}^{B}\left(X',X\right)}{1+\underline{\bar{k}}\left(X\right)}\bar{K}\left|\bar{\Psi}\left(\bar{K},X\right)\right|^{2}d\bar{K}dX$$

$$S_1^B(X',K',X) = \frac{K'\underline{k}_1^{(B)}(X',X)\left|\Psi(K',X')\right|^2}{1+\underline{k}(X')+\underline{k}_1^{(B)}(X')+\kappa\left[\frac{\underline{k}_2^{(B)}}{1+\underline{k}}\right](X')} \tag{143}$$

$$S_2^B(X',K',X) = \frac{\frac{\kappa\underline{k}_2^{(B)}(X',X)}{1+\underline{k}(X)}K'\left|\Psi(K',X')\right|^2}{1+\underline{k}(X')+\underline{k}_1^{(B)}(X')+\kappa\left[\frac{\underline{k}_2^{(B)}}{1+\underline{k}}\right](X')}$$

$$S_1^B(X',X) = \frac{K_{X'}\underline{k}_1^{(B)}(X',X)\left|\Psi(K',X')\right|^2}{1+\underline{k}(X')+\underline{k}_1^{(B)}(X')+\kappa\left[\frac{\underline{k}_2^{(B)}}{1+\underline{k}}\right](X')}\left|\Psi(X')\right|^2$$

$$S_2^B(X',X) = \frac{\frac{\kappa\underline{k}_2^{(B)}(X',X)}{1+\underline{k}(X)}K_{X'}\left|\Psi(X')\right|^2}{1+\underline{k}(X')+\underline{k}_1^{(B)}(X')+\kappa\left[\frac{\underline{k}_2^{(B)}}{1+\underline{k}}\right](X')}$$

for banks. This leads directly to:

$$\begin{aligned}\bar{M}\left(\left(\bar{K},X\right),\left(\bar{K}',X'\right)\right) &= \frac{\bar{\bar{k}}(X,X')\bar{K}}{1+\int\bar{\bar{k}}(X,X')\bar{K}_0'\left|\bar{\Psi}\left(\bar{K}_0',X'\right)\right|^2} \\ &\rightarrow \bar{S}\left(\left(\bar{K},X\right),\left(\bar{K}',X'\right)\right)\end{aligned}$$

$$\begin{aligned}&\hat{M}\left(\left(\hat{K}',X'\right),\left(\hat{K},X\right)\right) \\ =\ & \frac{\hat{k}(X,X')\hat{K}}{1+\int\hat{k}(X,X')\left|\hat{\Psi}\left(\hat{K}',X'\right)\right|^2+\int\hat{k}_1^B(X,X')\bar{K}_0'\left|\bar{\Psi}\left(\bar{K}_0',X'\right)\right|^2+\int\hat{k}_2^B(X,X')\frac{\bar{K}_0'\left|\bar{\Psi}\left(\bar{K}_0',X'\right)\right|^2}{1+\int\bar{\bar{k}}(X')\bar{K}_0''\left|\bar{\Psi}\left(\bar{K}_0'',X''\right)\right|^2}} \\ \rightarrow\ & \hat{S}\left(\left(\hat{K}',X'\right),\left(\hat{K},X\right)\right)\end{aligned}$$

and ultimately:

$$\begin{aligned}\bar{N} \rightarrow\ & \frac{\left(\hat{k}_1^B(X,X')+\kappa\frac{\hat{k}_2^B(X,X')}{1+\int\bar{\bar{k}}(X')\bar{K}_0''\left|\bar{\Psi}\left(\bar{K}_0'',X''\right)\right|^2}-\kappa\int\frac{\hat{k}_2^B(X,X'')\bar{K}_0''\bar{\bar{k}}(X'',X')}{\left(1+\int\bar{\bar{k}}\left(X'',\bar{Y}\right)\bar{K}_0^Y\left|\bar{\Psi}\left(\bar{K}_0^Y,\bar{Y}\right)\right|^2\right)^2}\right)\hat{K}}{1+\int\hat{k}(X,X')\left|\hat{\Psi}\left(\hat{K}',X'\right)\right|^2+\int\hat{k}_1^B(X,X')\bar{K}_0'\left|\bar{\Psi}\left(\bar{K}_0',X'\right)\right|^2+\kappa\int\hat{k}_2^B(X,X')\frac{\bar{K}_0'\left|\bar{\Psi}\left(\bar{K}_0',X'\right)\right|^2}{1+\int\bar{\bar{k}}(X')\bar{K}_0''\left|\bar{\Psi}\left(\bar{K}_0'',X''\right)\right|^2}} \\ \rightarrow\ & \hat{S}_1^B\left(\left(\hat{K},X\right),\left(\bar{K}',X'\right)\right)+\hat{S}_2^B\left(\left(\hat{K},X\right),\left(\bar{K}',X'\right)\right)\end{aligned}$$

A2.1.3 Returns equations as constraints

The return equations for investors and banks:

$$0=\int\left(\delta(X'-X)-\hat{S}_E(X',X)\right)\widehat{DF}(X')\hat{R}_{exc}(X')dX'-S_E(X,X)R_{exc}(X) \tag{144}$$

and:

$$\begin{aligned}0 &= \int\left(\delta(X'-X)-\bar{S}_E(X',X)\right)\overline{DF}(X')\bar{R}_{exc}(X') \\ &\quad -\int\hat{S}_E^B(X',X)\widehat{DF}(X')\hat{R}_{exc}(X')-S_E^B(X',X)R_{exc}(X)\end{aligned} \tag{145}$$

with:

$$\widehat{DF}(X)=\frac{1-\hat{S}(X)}{1-\hat{S}_E(X)}$$

$$\overline{DF}(X)=\frac{1-\bar{S}(X)}{1-\bar{S}_E(X)}$$

and:

$$\hat{R}_{exc}(X')=f(X')-\bar{r}$$

$$R_{exc}(X)=f(X)-\bar{r}$$

$$\bar{R}_{exc}(X')=\bar{f}(X')-(1+\kappa)\bar{r}$$

should be included in the action functional minimization equations through Lagrange multipliers $\bar{\xi}(X)$ and $\hat{\xi}(X)$ modifying these equations by terms of the form:

$$\begin{aligned}&\hat{\xi}(X)\int\left(\delta(X'-X)-\hat{S}_E(X',X)\right)\frac{\delta\widehat{DF}(X')}{\delta\hat{\Psi}\left(\hat{K},X\right)}\hat{R}_{exc}(X')\,dX'-\frac{\delta S_E(X,X)}{\delta\hat{\Psi}(K,X)}R_{exc}(X)\\&-\hat{\xi}(X)\hat{S}_E(X',X)\widehat{DF}(X')\hat{R}_{exc}(X')\,dX'\end{aligned}$$

and similar contributions for the derivatives with respects to $\bar{\Psi}\left(\bar{K},X\right)$.

We can neglect these contributions for two reasons. First, the stakes $S_E(X,X)$, $\hat{S}_E(X',X)$ $\bar{S}_E(X',X)$, $\hat{S}_E^B(X',X)$, $S_E^B(X',X)$ as well as $\widehat{DF}(X)$ and $\overline{DF}$ depend weakly on the fields $\hat{\Psi}\left(\hat{K},X\right)$ and $\bar{\Psi}\left(\bar{K},X\right)$, given that they correspond to decision of investments with respect to other agents level of return and uncertaint.

Second, these contributions are proportional to excess returns, so that they corresponds to corrections relatively to the levels of returns. We can thus neglect them in first approximation.

A2.1.4 Action functionals' second order derivatives

The second order derivaives for investors and fields action functionals are given by:

$$\begin{aligned}\frac{\delta^2}{\delta\Gamma\delta\Gamma^\dagger}S(\Gamma)\quad\simeq\quad&-\sum_\eta\left(\sigma_{\hat{K}}^2\nabla_{\hat{S}_\eta^{(T)}}^2-\left(\frac{\left(\hat{S}_\eta^{(T)}\right)^2}{2\hat{h}_\eta(X',X)}-\hat{V}_\eta\hat{S}_\eta^{(T)}\right)-\beta\right)-\underline{\lambda(X)}\left(\sum\hat{S}_\eta^{(T)}-1\right)\\&-\int\Gamma^\dagger\left(\hat{S}_\eta^{(T)}\right)\left[\frac{\delta^2}{\delta\Gamma\delta\Gamma^\dagger}\underline{\hat{V}_\eta}\hat{S}_\eta^{(T)}+\left(\frac{\delta}{\delta\Gamma}+\frac{\delta}{\delta\Gamma^\dagger}\right)\underline{\hat{V}_\eta}\hat{S}_\eta^{(T)}\left(\sum\hat{S}_\eta^{(T)}-1\right)\right]\Gamma\left(\hat{S}_\eta^{(T)}\right)\end{aligned}$$

and:

$$\begin{aligned}\frac{\delta^2}{\delta\bar{\Gamma}\delta\bar{\Gamma}^\dagger}S\left(\bar{\Gamma}\right)\quad=\quad&-\sum_\eta\left(\sigma_{\hat{K}}^2\nabla_{\bar{S}_\eta^{(T)}}^2-\left(\frac{\left(\bar{S}_\eta^{(T)}\right)^2}{2\hat{h}_\eta(X',X)}-\bar{V}_\eta\bar{S}_\eta^{(T)}\right)-\beta\right)-\underline{\lambda(X)}\left(\sum\bar{S}_\eta^{(T)}-1\right)\\&-\bar{\Gamma}\left(\bar{S}_\eta^{(T)}\right)\left[\frac{\delta^2}{\delta\Gamma\delta\Gamma^\dagger}\bar{V}_\eta\bar{S}_\eta^{(T)}+\left(\frac{\delta}{\delta\bar{\Gamma}}+\frac{\delta}{\delta\bar{\Gamma}^\dagger}\right)\bar{V}_\eta\bar{S}_\eta^{(T)}\left(\sum\bar{S}_\eta^{(T)}-1\right)\right]\bar{\Gamma}^\dagger\left(\bar{S}_\eta^{(T)}\right)\end{aligned}$$

where β represents the inverse time scale of the system, i.e. the inverse average persistence time of a collective state.

In the above expressions, the stakes $\hat{S}^{(T)}(X',X)$ and $\bar{S}^{(T)}(X',X)$ are respectively:

$$\hat{S}^{(T)}(X',X)=\left[\left\{\hat{S}_\eta(X',X)\right\}_{\eta=1,2},\left\{S_\eta(X,X)\right\}_{\eta=1,2}\right]$$

and:

$$\bar{S}^{(T)}(X',X)=\left[\left\{\bar{S}_{\eta}(X',X)\right\}_{\eta=1,2},\left\{\hat{S}_{\eta}(X',X)\right\}_{\eta=1,2},\left\{S_{\eta}(X,X)\right\}_{\eta=1,2}\right]$$

They are computed in the collective state defined by (106) and (108).

These formulas will be rewritten in a more compact form in the sequel of this work, in order to study the dynamics of deviations states.

A2.2 Fields of stakes' action functionals and minimization functions

A2.2.1 Micro model for investors stakes

We start with the sum over points of investors objective functions integrated over time:

$$\sum_{j}\hat{S}_{1ij}\hat{f}_{j}+\sum_{j}\hat{S}_{2ij}\hat{r}_{j}-\frac{1}{2}\sum_{ij}\frac{\left(\hat{S}_{\eta ij}\right)^{2}}{\hat{h}_{\eta_{i}}(X_{j})}+\sum_{k}S_{1ik}f_{k}+\sum_{k}S_{2ik}\bar{r}_{k}-\frac{1}{2}\sum_{ik}\frac{\left(S_{\eta ik}\right)^{2}}{h_{\eta ik}(X_{k})}$$

$$\int dt\left(\sum_{i,j,k}\hat{S}_{1ij}\hat{f}_{j}+\sum_{i,j,k}\hat{S}_{2ij}\hat{r}_{j}-\frac{1}{2}\sum_{i,j,k}\frac{\left(\hat{S}_{\eta ij}\right)^{2}}{\hat{h}_{\eta_{ij}}(X_{j})}+\sum_{i,k,j}S_{1ik}f_{k}+\sum_{i,k,j}S_{2ik}\bar{r}_{k}-\frac{1}{2}\sum_{ji,k}\frac{\left(S_{\eta ik}\right)^{2}}{h_{\eta ik}(X_{k})}\right)\tag{146}$$

Then, we add an inertial term for all agents and a term corresponding to the time scale in which collective states are considered:

$$-\alpha\sum_{i,j}\left(\frac{d}{dt}\hat{S}_{\eta ij}\right)^{2}-\alpha\sum_{i,k}\left(\frac{d}{dt}S_{\eta ik}\right)^{2}-\beta\int dt\tag{147}$$

A2.2.2 Field translation and investors stakes action functional

Considering the field:

$$\Gamma\left(S_{1},\hat{S}_{1},S_{2},\hat{S}_{2},X',X\right)$$

written:

$$\Gamma\left(\hat{S}^{(T)},X',X\right)$$

with:

$$\hat{S}^{(T)}=\left(S_{1},\hat{S}_{1},S_{2},\hat{S}_{2}\right)$$

the potential (146) becomes in terms of field:

$$-\sum_{\eta}\int\Gamma^{\dagger}\left(\hat{S}^{(T)},X',X\right)\left(-\frac{\left(\hat{S}_{\eta}^{(T)}\right)^{2}}{2\hat{h}_{\eta}^{T}(X',X)}+\hat{V}_{\eta}\hat{S}_{\eta}^{(T)}\right)\Gamma\left(\hat{S}^{(T)},X',X\right)d\left(\hat{S}^{(T)},X',X\right)$$

The coefficients $\hat{h}_{\eta}^{T}(X',X)$ translates the uncertainty coefficients $\hat{h}_{\eta_{i}}(X_{j})$, $h_{\eta ik}(X_{k})$ in inter, or intra, sectoral functions, while $\hat{V}_{\eta}$ encompass the returns. We have:

$$\begin{aligned}\hat{V}_{1}&=\hat{f}(X'),\hat{h}_{1}(X',X)=\hat{h}_{1}(X',X)\\\hat{V}_{2}&=\hat{r}(X'),\hat{h}_{2}(X',X)=\hat{h}_{2}(X',X)\end{aligned}$$

$$\begin{aligned}\hat{V}_{3}&=f(X),\hat{h}_{3}(X',X)=h_{1}(X,X)\\\hat{V}_{4}&=r(X),\hat{h}_{4}(X',X)=h_{2}(X,X)\end{aligned}$$

The contributions (147) are translated by:

$$-\sigma_{\hat{K}}^{2}\sum_{\eta}\int\Gamma^{\dagger}\left(\hat{S}^{(T)},X',X\right)\nabla_{\hat{S}_{\eta}^{(T)}}^{2}\Gamma\left(\hat{S}^{(T)},X',X\right)d\left(\hat{S}^{(T)},X',X\right)-\int\beta\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^{2}d\left(\hat{S}^{(T)},X',X\right)$$

We also add the contraint:

$$\int\lambda\left(X\right)\left(\sum_{\eta}\int\hat{S}_{\eta}^{(T)}\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^{2}dX'd\hat{S}^{(T)}-1\right)\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^{2}d\left(\hat{S}^{(T)},X',X\right)$$

The factor $\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^{2}$ accounts for the number of agents in sector X' connected to X.

The full action functional is the sum of these contributions:

$$\begin{aligned}&-\sigma_{\hat{K}}^{2}\sum_{\eta}\int\Gamma^{\dagger}\left(\hat{S}^{(T)},X',X\right)\nabla_{\hat{S}_{\eta}^{(T)}}^{2}\Gamma\left(\hat{S}^{(T)},X',X\right)d\left(\hat{S}^{(T)},X',X\right)-\int\beta\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^{2}d\left(\hat{S}^{(T)},X',X\right)\\&+\sum_{\eta}\int\left(\frac{\left(\hat{S}_{\eta}^{(T)}\right)^{2}}{2\hat{h}_{\eta}}-\hat{V}_{\eta}\hat{S}_{\eta}^{(T)}-\beta\right)\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^{2}d\left(\hat{S}^{(T)},X',X\right)\\&\int\lambda\left(X\right)\left(\sum_{\eta}\int\hat{S}_{\eta}^{(T)}\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^{2}dX'd\hat{S}^{(T)}-1\right)\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^{2}d\left(\hat{S}^{(T)},X',X\right)\end{aligned}$$

A2.2.3 Investors' stakes minimization equations and background field

The minimization equations is obtained by the derivative with respect to $\Gamma\left(\hat{S}^{(T)},X',X\right)$:

$$\begin{aligned}0&=\left(\sum_{\eta}\left(-\sigma_{\hat{K}}^{2}\nabla_{\hat{S}_{\eta}^{(T)}}^{2}+\frac{\left(\hat{S}_{\eta}^{(T)}\right)^{2}}{2\hat{h}_{\eta}}-\hat{V}_{\eta}\hat{S}_{\eta}^{(T)}\right)-\beta\right)\Gamma\left(\hat{S}^{(T)},X',X\right)\\&+\lambda\left(X\right)\left(\sum_{\eta}\int\hat{S}_{\eta}^{(T)}\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^{2}dX'd\hat{S}^{(T)}-1\right)\Gamma\left(\hat{S}^{(T)},X',X\right)\\&+\lambda\left(X\right)\Gamma\left(\hat{S}^{(T)},X',X\right)\hat{S}_{\eta}^{(T)}\left\|\Gamma\left(\hat{S}^{(T)},X',X\right)\right\|^{2}\end{aligned}\tag{148}$$

along with the derivative with respect to $\lambda\left(X\right)$ which implies:

$$\left(\sum_{\eta}\int\hat{S}_{\eta}^{(T)}\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^{2}dX'd\hat{S}^{(T)}-1\right)=0\tag{149}$$

and (148) simplifies as:

$$\begin{aligned}0&=\left(\sum_{\eta}\left(-\sigma_{\hat{K}}^{2}\nabla_{\hat{S}_{\eta}^{(T)}}^{2}+\frac{\left(\hat{S}_{\eta}^{(T)}\right)^{2}}{2\hat{h}_{\eta}}-\hat{V}_{\eta}\hat{S}_{\eta}^{(T)}+\lambda\left(X\right)\left\|\Gamma\left(\hat{S}^{(T)},X',X\right)\right\|_{X}^{2}\hat{S}_{\eta}^{(T)}\right)-\beta\right)\\&\times\Gamma\left(\hat{S}^{(T)},X',X\right)\end{aligned}\tag{150}$$

where:

$$\left\|\Gamma\left(\hat{S}^{(T)},X',X\right)\right\|_{X}^{2}=\int\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^{2}dX'd\hat{S}^{(T)}$$

We can rescale $\lambda(X)$ and replace:

$$\lambda(X)\left\|\Gamma\left(\hat{S}^{(T)},X',X\right)\right\|_X^2\rightarrow\lambda(X)$$

Solutions of this equation have the form:

$$\Gamma_{0,X',X}\left(\hat{S}^{(T)}\right)\Gamma(X',X)$$

The function $\Gamma_{0,X}\left(\hat{S}^{(T)}\right)$ is a parabolic cylinder function depending on β. To simplify, we can adjust the parameter β so that:

$$\Gamma_{0,X',X}\left(\hat{S}^{(T)}\right)=N\exp\left(-\sum_\eta\frac{\left(\hat{S}_\eta^{(T)}-\overline{\hat{S}_\eta^{(T)}}(X',X)\right)^2}{2\sigma_{\hat{K}}^2}\right)$$

with N a normalization factor. The equation for $\overline{\hat{S}^{(T)}}$ is the minimum of:

$$\frac{\left(\hat{S}_\eta^{(T)}\right)^2}{2\hat{h}_\eta(X',X)}-\hat{V}_\eta(X',X)\hat{S}_\eta^{(T)}-\beta+\lambda(X)\hat{S}_\eta^{(T)}$$

so that:

$$\overline{\hat{S}_\eta^{(T)}}(X',X)=\hat{h}_\eta(X',X)\left(\hat{V}_\eta(X',X)+\lambda(X)\right)\tag{151}$$

The average $\overline{\hat{S}^{(T)}}$ is also the average:

$$\overline{\hat{S}_\eta^{(T)}}(X',X)=\frac{\int\hat{S}_\eta^{(T)}\left|\Gamma_{0,X',X}\left(\hat{S}^{(T)}\right)\right|^2d\hat{S}^{(T)}}{\int\left|\Gamma_{0,X',X}\left(\hat{S}^{(T)}\right)\right|^2d\hat{S}^{(T)}}=\frac{\int\hat{S}_\eta^{(T)}\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^2d\hat{S}^{(T)}}{\int\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^2d\hat{S}^{(T)}}\tag{152}$$

and (159) writes:

$$\int\hat{S}_\eta^{(T)}\left|\Gamma_{0,X',X}\left(\hat{S}^{(T)}\right)\right|^2d\hat{S}^{(T)}=\hat{h}_\eta(X',X)\left(\hat{V}_\eta(X',X)+\lambda(X)\right)\int\left|\Gamma_{0,X',X}\left(\hat{S}^{(T)}\right)\right|^2d\hat{S}^{(T)}$$

Multiplying by $\Gamma(X',X)$, we integrate with respect to X' and sum over η and we find:

$$\begin{aligned}&\sum_\eta\int\hat{S}_\eta^{(T)}\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^2dX'd\hat{S}^{(T)}\\=&\sum_\eta\int\hat{h}_\eta(X',X)\hat{V}_\eta(X',X)\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^2dX'd\hat{S}^{(T)}+\lambda(X)\int\hat{h}_\eta(X',X)\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^2dX'd\hat{S}^{(T)}\end{aligned}$$

Then using the constraint (149) yields:

$$1=\sum_\eta\int\hat{h}_\eta(X',X)\hat{V}_\eta(X',X)\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^2dX'd\hat{S}^{(T)}+\lambda(X)\int\hat{h}_\eta(X',X)\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^2dX'd\hat{S}^{(T)}$$

so that:

$$\lambda(X)=\frac{1-\sum_\eta\int\hat{h}_\eta(X',X)\hat{V}_\eta(X',X)\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^2dX'd\hat{S}^{(T)}}{\sum_\eta\int\hat{h}_\eta(X',X)\left|\Gamma\left(\hat{S}^{(T)},X',X\right)\right|^2dX'd\hat{S}^{(T)}}$$

using the normalization over $\hat{S}^{(T)}$, this becomes:

$$\lambda(X)=\frac{1-\sum_{\eta}\int\hat{h}_{\eta}(X',X)\hat{V}_{\eta}(X',X)|\Gamma(X',X)|^{2}dX'}{\sum_{\eta}\int\hat{h}_{\eta}(X',X)|\Gamma(X',X)|^{2}dX'}$$

In the sequel we consider that the distribution $|\Gamma(X',X)|^2$ varies slowly, so that:

$$\lambda(X)=\frac{1-\sum_{\eta}\int\hat{h}_{\eta}(X',X)\hat{V}_{\eta}(X',X)dX'}{\sum_{\eta}\int\hat{h}_{\eta}(X',X)dX'}$$

Using the explicit form for $\hat{V}_{\eta}(X',X)$ leads to the expanded form of $\lambda(X)$:

$$\begin{aligned}&\lambda(X)\\ \rightarrow\quad&\frac{1-\int\hat{h}_{1}(X',X)\hat{f}(X')-\int\hat{h}_{2}(X',X)\hat{r}(X')-h_{1}(X,X)f(X)-h_{2}(X,X)\bar{r}}{\hat{h}_{1}(X)+\hat{h}_{2}(X)+h_{1}(X)+h_{2}(X)}\end{aligned}\tag{153}$$

where we set:

$$\hat{h}_{\eta}(X)=\int\hat{h}_{\eta}(X',X)dX'$$

In the sequel we will simplify the notation, and $\overline{\hat{S}_{\eta}^{(T)}}(X',X)\rightarrow\hat{S}_{\eta}^{(T)}(X',X)$, so that the average shares equations write:

$$\hat{S}_{\eta}^{(T)}(X',X)=\hat{h}_{\eta}(X',X)\left(\hat{V}_{\eta}(X',X)+\lambda(X)\right)\tag{154}$$

These minimizations equations for the shares will be studied in appendix 3.

Ultimately, remark that (160) also writes:

$$\hat{S}_{\eta}(X',X)=\int\hat{S}_{\eta}\left|\Gamma\left(S_{1},\hat{S}_{1},S_{2},\hat{S}_{2},X',X\right)\right|^{2}d\left(S_{1},\hat{S}_{1},S_{2},\hat{S}_{2}\right)$$

for the shares in investors, and:

$$S_{\eta}(X)=\int S_{\eta}\left|\Gamma\left(S_{1},\hat{S}_{1},S_{2},\hat{S}_{2},X',X\right)\right|^{2}d\left(S_{1},\hat{S}_{1},S_{2},\hat{S}_{2},X'\right)$$

for the shares in firms.

The constraint on stakes writes:

$$\int\left(\hat{S}_{1}(X',X)+\hat{S}_{2}(X',X)\right)dX'+\int\left(S_{1}(X',X)+S_{2}(X',X)\right)dX'=1$$

A2.2.4 Micro model for banks' stakes

Considering banks, we start with the full objective function for the system of banks:

$$\begin{aligned}\int dt\sum_{i}\Bigg(&\sum_{j}\bar{S}_{1ij}^{B}\bar{f}_{j}+\sum_{j}\bar{S}_{2ij}^{B}\bar{r}_{j}-\frac{1}{2}\sum_{j}\frac{\left(\bar{S}_{\eta ij}\right)^{2}}{\bar{h}_{\eta_{i}}(X_{j})}\\&+\sum_{j}\hat{S}_{1ij}^{B}\hat{f}_{j}-\frac{1}{2}\sum_{j}\frac{\left(\hat{S}_{1ij}^{B}\right)^{2}}{\hat{h}_{1i}(X_{j})}+\sum_{k}S_{1ik}^{B}f_{k}-\frac{1}{2}\sum_{k}\frac{\left(S_{1ik}^{B}\right)^{2}}{h_{\eta ik}(X_{k})}\Bigg)\end{aligned}\tag{155}$$

and for loans:

$$\int dt\sum_{i}\left(\sum_{j}\hat{S}_{2ij}^{B}\hat{r}_{j}-\frac{1}{2}\sum_{j}\frac{\left(\hat{S}_{2ij}^{B}\right)^{2}}{\hat{h}_{\eta 2i}(X_{j})}+\sum_{k}S_{2ik}^{B}\bar{r}_{k}-\frac{1}{2}\sum_{k}\frac{\left(S_{2ik}^{B}\right)^{2}}{h_{\eta ik}(X_{k})}\right)\tag{156}$$

A2.2.5 Field translation and banks's stakes action functional

Starting with banks shares fields:

$$\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)=\bar{\Gamma}\left(\bar{S}_1,S_1,\hat{S}_1,\bar{S}_2,S_2,\hat{S}_2,X',X\right)$$

The functions (155) and (156) are translated by the action functionals:

$$\sum_{\eta}\int\left(\frac{\left(\bar{S}_{\eta}^{T}\right)^{2}}{2\bar{h}_{\eta}^{T}\left(X',X\right)}-\bar{V}_{\eta}\bar{S}_{\eta}^{T}\right)\left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^{2}d\left(\bar{S}^{(T)},X',X\right)\tag{157}$$

where:

$$\bar{V}_1\left(X',X\right)=\bar{f}\left(X'\right),\ \bar{V}_2\left(X',X\right)=\bar{r}\left(X'\right)$$
$$\bar{V}_3\left(X',X\right)=\hat{f}\left(X'\right),\ \bar{V}_4\left(X',X\right)=\hat{r}\left(X'\right)$$
$$\bar{V}_5\left(X',X\right)=f\left(X\right),\ \bar{V}_6\left(X',X\right)=r\left(X\right)$$

and:

$$\bar{h}_1^{T}\left(X',X\right)=\bar{h}_1\left(X',X\right),\ \bar{h}_2^{T}\left(X',X\right)=\bar{h}_2\left(X',X\right)$$
$$\hat{h}_3^{T}\left(X',X\right)=\hat{h}_1\left(X',X\right),\ \bar{h}_4^{T}\left(X',X\right)=\hat{h}_2\left(X',X\right)$$
$$\hat{h}_5^{T}\left(X',X\right)=h_1\left(X,X\right),\ \bar{h}_6^{T}\left(X',X\right)=h_2\left(X,X\right)$$

As for investors, we add to the functional (157) a term accounting for inertia and a contribution characteristic of the time scale considered for the collective states:

$$-\sigma_{\hat{K}}^{2}\sum_{\eta}\int\bar{\Gamma}^{\dagger}\left(\bar{S}^{(T)},X',X\right)\nabla_{\bar{S}_{\eta}^{(T)}}^{2}\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)d\left(\bar{S}^{(T)},X',X\right)$$
$$-\int\beta\left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^{2}d\left(\bar{S}^{(T)},X',X\right)$$

Translating the constraints:

$$\bar{S}_{1ij}+\bar{S}_{2ij}+\hat{S}_{1ij}^{B}+S_{1ik}^{B}=1$$

and:

$$\hat{S}_{2ij}^{B}+S_{2ik}^{B}=\kappa\left(1-\sum_{j}\bar{S}_{2ij}^{B}\right)$$

leads to the two following contributions:

$$\begin{aligned}
&\int\lambda\left(X\right)\left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^{2}d\left(\bar{S}^{(T)},X',X\right)\\
&\times\left(\int\left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^{2}\left(\bar{S}_1d\left(\bar{S}_1,X'\right)+\bar{S}_2d\left(\bar{S}_2,X'\right)+\hat{S}_1^{B}d\left(\hat{S}_1^{B},X'\right)+S_1^{B}d\left(S_1^{B},X'\right)\right)-1\right)\\
&+\int\lambda'\left(X\right)\left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^{2}d\left(\bar{S}^{(T)},X',X\right)\\
&\times\left(\int\left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^{2}\left(\hat{S}_2^{B}d\left(\hat{S}_1^{B},X'\right)+S_2^{B}d\left(S_1^{B},X'\right)\right)-\kappa\left(1-\bar{S}_2^{B}\right)d\left(S_2^{B},X'\right)\right)\\
&+\int\lambda\left(X\right)\left(\sum_{\eta}\int\bar{S}_{\eta}\left(X',X\right)dX'+\int\hat{S}_1^{B}\left(X',X\right)dX'+S_1^{B}\left(X,X\right)-1\right)\left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^{2}\\
&+\int\lambda'\left(X\right)\left(\int\hat{S}_2^{B}\left(X',X\right)dX'+S_2^{B}\left(X,X\right)-\kappa\left(1-\bar{S}_{\eta}^{B}\left(X\right)\right)\right)\left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^{2}
\end{aligned}$$

we define the various averages in the states defined by the fields:

$$\begin{aligned}
\bar{S}_{\eta}^{B}\left(X',X\right) &= \int \bar{S}_{\eta}\left|\bar{\Gamma}\right|^{2} d\left(\bar{S}_{1},S_{1},\hat{S}_{1},\bar{S}_{2},S_{2},\hat{S}_{2},X'\right) \\
S_{\eta}^{B}\left(X,X\right) &= \int S_{\eta}^{B}\left|\bar{\Gamma}\right|^{2} d\left(\bar{S}_{1},S_{1},\hat{S}_{1},\bar{S}_{2},S_{2},\hat{S}_{2},X'\right) \\
\hat{S}_{\eta}^{B}\left(X',X\right) &= \int \hat{S}_{\eta}^{B}\left|\bar{\Gamma}\right|^{2} d\left(\bar{S}_{1},S_{1},\hat{S}_{1},\bar{S}_{2},S_{2},\hat{S}_{2},X'\right)
\end{aligned}$$

We also define:

$$\bar{S}^{(T)}\left(X',X\right)\left|\Gamma\left(X',X\right)\right|^{2} = \int \bar{S}^{(T)}\left|\Gamma\left(\bar{S},X',X\right)\right|^{2}$$

$$\begin{aligned}
\bar{S}_{\eta}^{B}\left(X\right) &= \int \bar{S}_{\eta}^{B}\left(X',X\right)dX' \\
\hat{S}_{\eta}^{B}\left(X\right) &= \int \hat{S}_{\eta}^{B}\left(X',X\right)dX'
\end{aligned}$$

and the constraints on allocation in this context writes:

$$\sum_{\eta=1}^{2}\int \bar{S}_{\eta}^{B}\left(X',X\right)dX' + \int \hat{S}_{1}^{B}\left(X',X\right)dX' + S_{1}^{B}\left(X,X\right) = 1$$

$$\int \hat{S}_{2}^{B}\left(X',X\right)dX' + S_{2}^{B}\left(X,X\right) = \kappa\left(1-\bar{S}_{2}^{B}\left(X\right)\right)$$

As a consequence, the contraint rewrite in terms of fields:

$$\begin{aligned}
&\int \lambda\left(X\right)\left(\sum_{\eta=1}^{2}\int \bar{S}_{\eta}^{B}\left(X',X\right)dX' + \int \hat{S}_{1}^{B}\left(X',X\right)dX' + S_{1}^{B}\left(X,X\right)-1\right)\left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^{2} \\
&+\int \lambda'\left(X\right)\left(\int \hat{S}_{2}^{B}\left(X',X\right)dX' + S_{2}^{B}\left(X,X\right)-\kappa\left(1-\bar{S}_{\eta}^{B}\left(X\right)\right)\right)\left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^{2}
\end{aligned}$$

The action functional for shars is ths defined by:

$$\begin{aligned}
S\left(\bar{\Gamma}\right) &= -\sigma_{\hat{K}}^{2}\sum_{\eta}\int \bar{\Gamma}^{\dagger}\left(\bar{S}^{(T)},X',X\right)\nabla_{\bar{S}_{\eta}^{(T)}}^{2}\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)d\left(\bar{S}^{(T)},X',X\right) \\
&+\int\left(\sum_{\eta}\left(\frac{\left(\bar{S}_{\eta}^{(T)}\right)^{2}}{2\hat{h}_{\eta}\left(X',X\right)}-\bar{V}_{\eta}\bar{S}_{\eta}^{(T)}\right)-\beta\right)\left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^{2}d\left(\bar{S}^{(T)},X',X\right) \\
&+\int \lambda\left(X\right)\left(\sum_{\eta=1}^{2}\int \bar{S}_{\eta}^{B}\left(X',X\right)dX' + \int \hat{S}_{1}^{B}\left(X',X\right)dX' + S_{1}^{B}\left(X,X\right)-1\right)\left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^{2} \\
&+\int \lambda'\left(X\right)\left(\int \hat{S}_{2}^{B}\left(X',X\right)dX' + S_{2}^{B}\left(X,X\right)-\kappa\left(1-\bar{S}_{\eta}^{B}\left(X\right)\right)\right)\left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^{2}
\end{aligned} \tag{158}$$

$$\bar{S}^{(T)} = \left[\bar{S}_{1},S_{1},\hat{S}_{1},\bar{S}_{2},S_{2},\hat{S}_{2}\right]$$

A2.2.6 Banks' stakes minimization equations and background field

The minimization equations for investors has been presented above (see (148)). The minimization equations for (158) are similar. They are obtained by using the derivatives with respect to $\Gamma\left(\hat{S}^{(T)},X',X\right)$, $\lambda(X)$ and $\lambda'(X)$. These two derivatives implement the constraints. We find:

$$-\sigma_{\hat{K}}^{2}\sum_{\eta}\nabla_{\bar{S}_{\eta}^{(T)}}^{2}\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)$$
$$+\left(\sum_{\eta}\left(\frac{\left(\bar{S}_{\eta}^{(T)}\right)^{2}}{2\bar{w}_{\eta}(X',X)}-\bar{V}_{\eta}^{(T)}\bar{S}_{\eta}^{T}+\lambda_{\eta}(X)\bar{S}_{\eta}^{T}\right)-\beta\right)\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)$$

with:

$$\begin{aligned}\lambda_{\eta}(X)&=\lambda(X)\text{ for }\eta=1,2,3,5\\\lambda_{\eta}(X)&=\lambda'(X)\text{ for }\eta=5,6\end{aligned}$$

and where we rescaled:

$$\begin{aligned}\lambda(X)\left\|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right\|_{X}^{2}&\rightarrow\lambda(X)\\\lambda'(X)\left\|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right\|_{X}^{2}&\rightarrow\lambda'(X)\end{aligned}$$

with:

$$\left\|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right\|_{X}^{2}=\int\left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^{2}d(X')\,d\bar{S}^{(T)}$$

As in Gosselin and Lotz (2025), we can adjust β, so that the solution to the minimization equations have the form:

$$\Gamma_{0,X',X}\left(\bar{S}^{(T)}\right)\Gamma(X',X)$$

where:

$$\Gamma_{0,X',X}\left(\bar{S}^{(T)}\right)=N\exp\left(-\sum_{\eta}\frac{\left(\bar{S}_{\eta}^{(T)}-\overline{\bar{S}_{\eta}^{(T)}}(X',X)\right)^{2}}{2\sigma_{\hat{K}}^{2}}\right)$$

and the average $\overline{\bar{S}_{\eta}^{(T)}}$ is given by:

$$\overline{\bar{S}_{\eta}^{(T)}}(X',X)=\bar{w}_{\eta}(X',X)\left(\bar{V}_{\eta}(X',X)+\lambda_{\eta}(X)\right)\tag{159}$$

The average $\overline{\hat{S}^{(T)}}$ s also the average:

$$\begin{aligned}\overline{\bar{S}_{\eta}^{(T)}}(X',X)&=\frac{\int\bar{S}_{\eta}^{(T)}\left|\Gamma_{0,X',X}\left(\bar{S}^{(T)}\right)\right|^{2}d\bar{S}^{(T)}}{\int\left|\Gamma_{0,X',X}\left(\bar{S}^{(T)}\right)\right|^{2}d\bar{S}^{(T)}}\\&=\frac{\int\bar{S}_{\eta}^{(T)}\left|\Gamma\left(\bar{S}^{(T)},X',X\right)\right|^{2}d\bar{S}^{(T)}}{\int\left|\Gamma\left(\bar{S}^{(T)},X',X\right)\right|^{2}d\bar{S}^{(T)}}\end{aligned}\tag{160}$$

Multiplying by $\Gamma(X',X)$, we integrate with respect to X':

$$\int\bar{S}_{\eta}^{(T)}\left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^{2}d(X')\,d\bar{S}^{(T)}$$
$$=\int\bar{w}_{\eta}(X',X)\bar{V}_{\eta}\left(\bar{S}^{(T)},X',X\right)\left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^{2}d(X')\,d\bar{S}^{(T)}$$
$$+\lambda_{\eta}(X)\int\bar{w}_{\eta}(X',X)\left|\bar{\Gamma}(X',X)\right|^{2}d(X')\,d\bar{S}^{(T)}$$

Then the partial summation over η and the constraints yield:

$$\begin{aligned}
1 &= \sum_{\eta\in\{1,2,3,5,\}} \int \bar{w}_\eta (X',X) \bar{V}_\eta (X',X) \left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^2 d(X') d\bar{S}^{(T)} \\
&\quad +\lambda(X) \sum_{\eta\in\{1,2,3,5,\}} \int \bar{w}_\eta (X',X) \left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^2 d(X') d\bar{S}^{(T)}
\end{aligned}$$

$$\begin{aligned}
&\kappa\left(1-\bar{S}_L^B(X)\right) \\
&= \sum_{\eta\in\{4,6\}} \int \bar{w}_\eta (X',X) \bar{V}_\eta (X',X) \left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^2 d(X') d\bar{S}^{(T)} \\
&\quad +\lambda'(X) \sum_{\eta\in\{4,6\}} \int \bar{w}_\eta (X',X) \left|\bar{\Gamma}\left(\bar{S}^{(T)},X',X\right)\right|^2 d(X') d\bar{S}^{(T)}
\end{aligned}$$

Using the normalization over $\hat{S}^{(T)}$, we find the Lagrange multipliers:

$$\lambda(X) = \frac{1-\sum_{\eta\in\{1,2,3,5,\}} \int \bar{w}_\eta (X',X) \bar{V}_\eta (X',X) \left|\bar{\Gamma}(X',X)\right|^2 d(X')}{\sum_{\eta\in\{1,2,3,5,\}} \int \bar{w}_\eta (X',X) \bar{V}_\eta (X',X) \left|\bar{\Gamma}(X',X)\right|^2 d(X')}$$

$$\lambda'(X) = \frac{\kappa\left(1-\bar{S}_L^B(X)\right) - \sum_{\eta\in\{4,6\}} \int \bar{w}_\eta (X',X) \bar{V}_\eta (X',X) \left|\bar{\Gamma}(X',X)\right|^2 d(X') d\bar{S}^{(T)}}{\sum_{\eta\in\{4,6\}} \int \bar{w}_\eta (X',X) \left|\bar{\Gamma}(X',X)\right|^2 d(X') d\bar{S}^{(T)}}$$

In the sequel we consider that the distribution $|\Gamma(X',X)|^2$ varies slowly, so that, using the explicit form for $\hat{V}_\eta(X',X)$ leads to the expanded form of $\lambda(X)$ and $\lambda'(X)$:

$$\lambda(X) = \frac{1-\int \bar{w}_E (X',X) \bar{f}(X') - \int \bar{w}_L (X',X) \bar{r}(X') - \int \hat{w}_E^B (X',X) \hat{f}(X') - w_E^B(X,X) f(X)}{\bar{w}_E(X) + \bar{w}_E(X) + \hat{w}_E^B(X) + w_E^B(X)} \tag{161}$$

and:

$$\lambda'(X) = \frac{\kappa\left(1-\bar{S}_L^B(X)\right) - \int \hat{w}_L (X',X) \hat{r}(X') - w_L(X,X)\bar{r}}{\hat{w}_L(X) + w_L(X)} \tag{162}$$

In the sequel, we will replace:

$$\overline{\bar{S}_\eta^{(T)}}(X',X) \to \bar{S}_\eta^{(T)}(X',X)$$

Appendix 3 Solutions in terms of stakes

A3.1 Formulas for shares between two sectors

A3.1.1 Investors

For investors' field, the saddle point equations are the same as in the first part. Collective states satisfy the saddle point equations for $S(\Gamma)$. The expanded form of equation (154) is:

$$\begin{aligned}
\hat{S}_E(X',X) &= \hat{w}_E(X',X)\left(\hat{f}(X') + \lambda(X)\right) \\
\hat{S}_L(X',X) &= \hat{w}_L(X',X)\left(\bar{r}(X',X) + \lambda(X)\right)
\end{aligned}$$

for investors-investors shares, and:

$$\begin{aligned} S_E(X,X) &= w_E(X)(f(X)+\lambda(X)) \\ S_L(X,X) &= w_E(X)(\bar{r}(X,X)+\lambda(X)) \end{aligned}$$

for investors-firms shares. The Lagrange multiplier $\lambda(X)$ was given in (161):

$$\begin{aligned} &\lambda(X) \\ \rightarrow\ & \frac{1-\int \hat{w}_E(X',X)\hat{f}(X')-\int \hat{w}_L(X',X)\hat{r}(X')-w_E(X,X)f(X)-w_L(X,X)\bar{r}}{\hat{w}_E(X)+\hat{w}_L(X)+w_E(X)+w_L(X)} \end{aligned}$$

where we set:

$$\hat{w}_\eta(X)=\int \hat{w}_\eta(X',X)$$

The solutions for shares are:

$$\begin{aligned} &\hat{S}_E(X',X) \\ =\ & \underline{\hat{S}}_E(X',X)+\frac{\hat{w}_E(X',X)}{\hat{w}_E(X)+\hat{w}_L(X)+w_E(X)+w_L(X)} \\ &\times\left\{\hat{w}_E(X)\left(\hat{f}(X')-\left\langle\hat{f}(X')\right\rangle_{\hat{w}_E}\right)+\hat{w}_L(X)\left(\hat{f}(X')-\langle\hat{r}(X')\rangle_{\hat{w}_L}\right)\right. \\ &\left.+w_E(X)\left(\hat{f}(X')-f(X)\right)+w_L(X)\left(\hat{f}(X')-\bar{r}\right)\right\} \end{aligned}$$

$$\begin{aligned} &\hat{S}_E(X',X) \\ =\ & \underline{\hat{S}}_E(X',X)+\frac{\hat{w}_E(X',X)}{\hat{w}_E(X)+\hat{w}_L(X)+w_E(X)+w_L(X)} \\ &\times\left\{\hat{w}_E(X)\left(\hat{f}(X')-\left\langle\hat{f}(X')\right\rangle_{\hat{w}_E}\right)+\hat{w}_L(X)\left(\hat{f}(X')-\langle\hat{r}(X')\rangle_{\hat{w}_L}\right)\right. \\ &\left.+w_E(X)\left(\hat{f}(X')-f(X)\right)+w_L(X)\left(\hat{f}(X')-\bar{r}\right)\right\} \end{aligned}$$

$$\begin{aligned} \hat{S}_L(X',X) =\ & \underline{\hat{S}}_L(X',X)+\frac{\hat{w}_L(X',X)}{\hat{w}_E(X)+\hat{w}_L(X)+w_E(X)+w_L(X)} \\ &\times\left\{\hat{w}_E(X)\left(\hat{r}(X')-\left\langle\hat{f}(X')\right\rangle_{\hat{w}_E}\right)+\hat{w}_L(X)\left(\hat{r}(X')-\langle\hat{r}(X')\rangle_{\hat{w}_L}\right)\right. \\ &\left.+w_E(X)(\hat{r}(X')-f(X))+w_L(X)(\hat{r}(X')-\bar{r})\right\} \end{aligned}$$

with:

$$\langle F(X')\rangle_{\hat{w}_\eta}=\frac{\int F(X')\hat{w}_\eta(X',X)}{\hat{w}_\eta(X)}$$

for any function $F(X')$ and:

$$\underline{\hat{S}}_\eta(X',X)=\frac{\hat{w}_\eta(X',X)}{\hat{w}_E(X)+\hat{w}_L(X)+w_E(X)+w_L(X)}$$

The shares of investment in firms are given by:

$$\begin{aligned} &S_E(X,X) \\ =\ & \underline{S}_E(X,X)+w_E(X) \\ &\times\frac{\hat{w}_E(X)\left(f(X)-\left\langle\hat{f}(X')\right\rangle_{\hat{w}_E}\right)+\hat{w}_L(X)\left(f(X)-\langle\hat{r}(X')\rangle_{\hat{w}_L}\right)+w_L(X)(f(X)-\bar{r})}{\hat{w}_E(X)+\hat{w}_L(X)+w_E(X)+w_L(X)} \end{aligned}$$

for participations, and:

$$\begin{aligned}
& S_L(X,X) \\
= \; & \underline{S}_L(X,X) + w_L(X) \\
& \times \frac{\hat{w}_E(X)\left(\bar{r}(X) - \left\langle \hat{f}(X') \right\rangle_{\hat{w}_E}\right) + \hat{w}_L(X)\left(\bar{r}(X) - \langle \hat{r}(X') \rangle_{\hat{w}_L}\right) + w_L(X)\left(\bar{r}(X) - f(X)\right)}{\hat{w}_E(X) + \hat{w}_L(X) + w_E(X) + w_L(X)}
\end{aligned}$$

for loans, with:

$$\underline{S}_\eta(X,X) = \frac{w_\eta(X',X)}{\int \hat{w}_E(X',X) + \int \hat{w}_L(X',X) + w_E(X) + w_L(X)}$$

The coefficients $\hat{w}_\alpha$, w_α are endogeneous since they depend on the uncertainty on returns depending themselves on th $\underline{\hat{S}}_\eta$ S_η. Solving will be done by detailling this uncertainties. Before doing so, we simplify slightly by imposing several assmptions.

First, we normalize $\hat{w}_E(X) + \hat{w}_L(X) + w_E(X) + w_L(X) = 1$ and choose:

$$\hat{w}_E(X',X) = \hat{w}_L(X',X) = \frac{1}{2}\hat{w}(X',X)$$

$$w_E(X,X) = w_L(X,X) = \frac{w(X,X)}{2}$$

$$\hat{w}_E(X) = \hat{w}_L(X) = \frac{\hat{w}(X)}{2}$$

$$w_E(X) = w_L(X) = \frac{w(X)}{2}$$

The constraint on coefficients implies:

$$w(X) = 1 - \hat{w}(X)$$

and the solutions writes ultimately:

$$\begin{aligned}
& \hat{S}_E(X',X) \\
= \; & \frac{\underline{\hat{S}}(X',X)}{2} + \frac{\hat{w}(X',X)}{2}\left(\hat{w}(X)\left(\hat{f}(X') - \frac{\left\langle \hat{f}(X') \right\rangle_{\hat{w}_E} + \langle \hat{r}(X') \rangle_{\hat{w}_L}}{2}\right) + w(X)\left(\hat{f}(X') - \frac{f(X) + r(X)}{2}\right)\right) \\
= \; & \frac{\underline{\hat{S}}(X',X)}{2} + \frac{\hat{w}(X',X)}{2}\left(\hat{f}(X') - \hat{w}(X)\frac{\left\langle \hat{f}(X') \right\rangle_{\hat{w}_E} + \langle \hat{r}(X') \rangle_{\hat{w}_L}}{2} - w(X)\frac{f(X) + r(X)}{2}\right)
\end{aligned}$$

$$\begin{aligned}
& \hat{S}_L(X',X) \\
= \; & \frac{\underline{\hat{S}}(X',X)}{2} + \frac{\hat{w}(X',X)}{2}\left(\hat{w}(X)\left(\hat{r}(X') - \frac{\left\langle \hat{f}(X') \right\rangle_{\hat{w}_E} + \langle \hat{r}(X') \rangle_{\hat{w}_L}}{2}\right) + w(X)\left(\hat{r}(X') - \frac{f(X) + r(X)}{2}\right)\right) \\
= \; & \frac{\underline{\hat{S}}(X',X)}{2} + \frac{\hat{w}(X',X)}{2}\left(\hat{r}(X') - \hat{w}(X)\frac{\left\langle \hat{f}(X') \right\rangle_{\hat{w}_E} + \langle \hat{r}(X') \rangle_{\hat{w}_L}}{2} - w(X)\frac{f(X) + r(X)}{2}\right)
\end{aligned}$$

with:

$$\underline{\hat{S}}_E(X',X) = \underline{\hat{S}}_L(X',X) = \frac{\underline{\hat{S}}(X',X)}{2} = \frac{1}{2}\hat{w}(X',X)$$

$$\hat{S}(X',X)$$
$$= \underline{\hat{S}}(X',X) + \hat{w}(X',X)\left(\frac{\hat{f}(X')+\hat{r}(X')}{2} - \hat{w}(X)\frac{\left\langle \hat{f}(X')\right\rangle_{\hat{w}_E} + \langle \hat{r}(X')\rangle_{\hat{w}_L}}{2} - w(X)\frac{f(X)+r(X)}{2}\right)$$

$$S_E(X,X) \tag{163}$$
$$= \frac{\underline{S}(X,X)}{2} + \frac{w(X)}{2}\left(\hat{w}(X)\left(f(X) - \frac{\left\langle \hat{f}(X')\right\rangle_{\hat{w}_E} + \langle \hat{r}(X')\rangle_{\hat{w}_L}}{2}\right) + \frac{w(X)}{2}\left(f(X) - \bar{r}(X)\right)\right)$$
$$= \frac{\underline{S}(X,X)}{2} + \frac{w(X)}{2}\left(f(X) - \hat{w}(X)\frac{\left\langle \hat{f}(X')\right\rangle_{\hat{w}_E} + \langle \hat{r}(X')\rangle_{\hat{w}_L}}{2} - w(X)\frac{f(X)+\bar{r}(X)}{2}\right) \tag{164}$$

$$S_L(X,X)$$
$$= \frac{\underline{S}(X,X)}{2} + \frac{w(X)}{2}\left(\hat{w}(X)\left(\bar{r}(X) - \frac{\left\langle \hat{f}(X')\right\rangle_{\hat{w}_E} + \langle \hat{r}(X')\rangle_{\hat{w}_L}}{2}\right) + \frac{w(X)}{2}\left(\bar{r}(X) - f(X)\right)\right)$$
$$= \frac{\underline{S}(X,X)}{2} + \frac{w(X)}{2}\left(\bar{r}(X) - \hat{w}(X)\frac{\left\langle \hat{f}(X')\right\rangle_{\hat{w}_E} + \langle \hat{r}(X')\rangle_{\hat{w}_L}}{2} - w(X)\frac{f(X)+\bar{r}(X)}{2}\right)$$

with:

$$\underline{S}_E(X,X) = \underline{S}_L(X,X) = \frac{\underline{S}(X,X)}{2} = \frac{w(X,X)}{2}$$
$$S(X,X) = \underline{S}(X,X) + w(X)\left(\hat{w}(X)\left(\frac{f(X)+\bar{r}(X)}{2} - \frac{\left\langle \hat{f}(X')\right\rangle_{\hat{w}_E} + \langle \hat{r}(X')\rangle_{\hat{w}_L}}{2}\right)\right) \tag{165}$$

A.3.1.2 Banks

For banks, they write:

$$\begin{aligned}
\bar{S}_E(X',X) &= \bar{w}_E(X',X)\left(\bar{f}(X') + \lambda(X)\right) \\
\bar{S}_L(X',X) &= \bar{w}_L(X',X)\left(\bar{r}(X') + \lambda(X)\right)
\end{aligned}$$

$$\begin{aligned}
\hat{S}_E^B(X',X) &= \hat{w}_E^B(X',X)\left(\hat{f}(X') + \lambda(X)\right) \\
\hat{S}_L^B(X',X) &= \hat{w}_L^B(X',X)\left(\hat{r}(X') + \lambda'(X)\right)
\end{aligned}$$

$$\begin{aligned}
S_E^B(X,X) &= w_E^B(X)\left(f(X) + \lambda(X)\right) \\
S_L^B(X,X) &= w_L^B(X)\left(r(X) + \lambda'(X)\right)
\end{aligned}$$

and the multiplers satisfy:

$$\lambda(X)$$
$$= \frac{1 - \int \bar{w}_E(X',X)\hat{f}(X') - \int \bar{w}_L(X',X)\hat{r}(X') - \int \hat{w}_E^B(X',X)\hat{f}(X') - w_E^B(X,X)f(X)}{\bar{w}_E(X) + \bar{w}_L(X) + \hat{w}_E^B(X) + w_E^B(X)}$$

and:

$$\begin{aligned}
&\lambda'(X)\\
=\quad&\frac{\kappa\left(1-\bar{S}(X)\right)-\int\hat{w}_L^B(X',X)\,\hat{r}(X')-w_L^B(X,X)\,\bar{r}}{\hat{w}_L^B(X)+w_L^B(X)}
\end{aligned}$$

we define the average coefficients:

$$\bar{w}_\eta(X)=\int\bar{w}_\eta(X',X)$$

and:

$$\hat{w}_\eta^B(X)=\int\hat{w}_\eta^B(X',X)$$

so that $\bar{S}_E(X',X)$ and $\bar{S}_L(X',X)$ are given by:

$$\begin{aligned}
&\bar{S}_E(X',X)\\
=\quad&\bar{S}_E(X',X)+\frac{\bar{w}_E(X',X)}{\bar{w}_E(X)+\bar{w}_L(X)+\hat{w}_E^B(X)+w_E^B(X)}\\
&\times\left\{\bar{w}_E(X)\left(\bar{f}(X')-\left\langle\bar{f}(X')\right\rangle_{\bar{w}_E}\right)+\bar{w}_L(X)\left(\bar{f}(X')-\left\langle\bar{r}(X')\right\rangle_{\bar{w}_L}\right)\right.\\
&\left.+\hat{w}_E^B(X)\left(\bar{f}(X')-\left\langle\hat{f}(X')\right\rangle_{\hat{w}_E}\right)+w_E^B(X)\left(\bar{f}(X')-f(X)\right)\right\}
\end{aligned}$$

$$\begin{aligned}
&\bar{S}_L(X',X)\\
=\quad&\bar{S}_L(X',X)+\frac{\bar{w}_L(X',X)}{\bar{w}_E(X)+\bar{w}_L(X)+\hat{w}_E^B(X)+w_E^B(X)}\\
&\times\left\{\bar{w}_E(X)\left(\bar{r}(X')-\left\langle\bar{f}(X')\right\rangle_{\bar{w}_E}\right)+\bar{w}_L(X)\left(\bar{r}(X')-\left\langle\bar{r}(X')\right\rangle_{\bar{w}_L}\right)\right.\\
&\left.+\hat{w}_E^B(X)\left(\bar{r}(X')-\left\langle\hat{f}(X')\right\rangle_{\hat{w}_E^B}\right)+w_E^B(X)\left(\bar{r}(X')-f(X)\right)\right\}
\end{aligned}$$

Defining the weighted averages of a function by:

$$\begin{aligned}
\left\langle F(X')\right\rangle_{\bar{w}_\eta}&=\frac{\int F(X')\,\bar{w}_\eta(X',X)}{\bar{w}_\eta(X)}\\
\left\langle F(X')\right\rangle_{\hat{w}_\eta}&=\frac{\int F(X')\,\hat{w}_\eta^B(X',X)}{\hat{w}_\eta^B(X)}
\end{aligned}$$

for any functions $F(X')$, $F(X')$ and:

$$\underline{\bar{S}}_\eta(X',X)=\frac{\bar{w}_\eta(X',X)}{\bar{w}_E(X)+\bar{w}_L(X)+\hat{w}_E^B(X)+w_E^B(X)}$$

the shares $\hat{S}_E^B(X',X)$ and $\hat{S}_L^B(X',X)$ are given by:

$$\begin{aligned}
&\hat{S}_E^B(X',X)\\
=\quad&\underline{\hat{S}}_E^B(X',X)+\frac{\hat{w}_E^B(X',X)}{\bar{w}_E(X)+\bar{w}_L(X)+\hat{w}_E^B(X)+w_E^B(X)}\\
&\times\left\{\bar{w}_E(X)\left(\hat{f}(X')-\left\langle\bar{f}(X')\right\rangle_{\bar{w}_E}\right)+\bar{w}_L(X)\left(\hat{f}(X')-\left\langle\bar{r}(X')\right\rangle_{\bar{w}_L}\right)\right.\\
&\left.+\hat{w}_E^B(X)\left(\hat{f}(X')-\left\langle\hat{f}(X')\right\rangle_{\hat{w}_E}\right)+w_E^B(X)\left(\hat{f}(X')-f(X)\right)\right\}
\end{aligned}$$

and:

$$\begin{aligned}
&\frac{\hat{S}_L^B(X',X)}{\kappa\left(1-\bar{S}(X)\right)}=\underline{\hat{S}}_L^B(X',X)\\
&+\frac{\hat{w}_L^B(X',X)}{\hat{w}_L^B(X)+w_L^B(X)}\left[\hat{w}_L^B(X)\left(\hat{r}(X')-\langle\hat{r}(X')\rangle_{\hat{w}_E}\right)+w_L^B(X)\left(\hat{r}(X')-\langle r(X')\rangle\right)\right]\\
\simeq\;&\underline{\hat{S}}_L^B(X',X)+\frac{\hat{w}_L^B(X',X)}{\hat{w}_L^B(X)+w_L^B(X)}w_L^B(X)\left(\hat{r}(X')-\langle r(X')\rangle\right)
\end{aligned}$$

where:

$$\underline{\hat{S}}_E(X',X)=\frac{\hat{w}_E^B(X',X)}{\bar{w}_E(X)+\bar{w}_L(X)+\hat{w}_E^B(X)+w_E^B(X)}$$

$$\underline{\hat{S}}_L^B(X',X)=\frac{\hat{w}_L^B(X',X)}{\hat{w}_L^B(X)+w_L^B(X)}$$

The invested shares in firms are given by:

$$\begin{aligned}
&S_E^B(X,X)\\
=\;&\underline{S}_E^B(X,X)+\frac{w_E^B(X)}{\bar{w}_E(X)+\bar{w}_L(X)+\hat{w}_E^B(X)+w_E^B(X)}\\
&\times\left\{\bar{w}_E(X)\left(f(X)-\left\langle\bar{f}(X')\right\rangle_{\bar{w}_E}\right)+\bar{w}_L(X)\left(f(X)-\langle\bar{r}(X')\rangle_{\bar{w}_L}\right)\right.\\
&\left.+\hat{w}_E^B(X)\left(f(X)-\left\langle\hat{f}(X')\right\rangle_{\hat{w}_E}\right)\right\}
\end{aligned}$$

$$\begin{aligned}
&\frac{S_L^B(X,X)}{\kappa\left(1-\bar{S}(X)\right)}\\
=\;&\underline{S}_L^B(X,X)+\frac{w_L^B(X)}{\hat{w}_L^B(X)+w_L^B(X)}\left(\hat{w}_L^B(X)\left(r(X)-\langle\hat{r}(X')\rangle_{\hat{w}_L}\right)+w_L^B(X)\left(r(X)-\langle r(X)\rangle\right)\right)\\
\simeq\;&\underline{S}_L^B(X,X)+\frac{w_L^B(X)}{\hat{w}_L^B(X)+w_L^B(X)}\hat{w}_L^B(X)\left(r(X)-\langle\hat{r}(X')\rangle_{\hat{w}_L}\right)
\end{aligned}$$

with:

$$\underline{S}_E^B(X,X)=\frac{w_E^B(X,X)}{\bar{w}_E(X)+\bar{w}_L(X)+\hat{w}_E^B(X)+w_E^B(X)}$$

$$\underline{S}_L^B(X,X)=\frac{w_L^B(X,X)}{\hat{w}_L^B(X)+w_L^B(X)}$$

The coefficients $\hat{w}_\alpha$, w_α are endogeneous since they depend on the uncertainty on returns depending themselves on the $\underline{\hat{S}}_\eta$ S_η. Solving will be done by detailling this uncertainties. Before doing so, we simplify slightly by imposing several assumptions.

First, we normalize:

$$\bar{w}_E(X)+\bar{w}_L(X)+\hat{w}_E^B(X)+w_E^B(X)=1$$

$$\hat{w}_L^B(X)+w_L^B(X)=1$$

and choose for the sake of simplicity:

$$\bar{w}_E(X',X)=\bar{w}_L(X',X)=\frac{1}{2}\bar{w}(X',X)$$

which describes that the perceived uncertainty for participations and loans are equal. This implies that:

$$\underline{\bar{S}}_E(X',X)=\underline{\bar{S}}_L(X',X)=\frac{\underline{\bar{S}}(X',X)}{2}=\frac{1}{2}\bar{w}(X',X)$$

Then defining:

$$\begin{aligned}\bar{S}_\eta(X') &= \int \bar{S}_\eta(X',X)\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_{X'}\left|\bar{\Psi}(X')\right|^2}dX \\ &\simeq \int \bar{S}_\eta(X',X)\,dX\frac{\langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^2}{\bar{K}_{X'}\left|\bar{\Psi}(X')\right|^2}\end{aligned}$$

$$\bar{w}(X')=\int\bar{w}(X',X)\,dX$$

$$\begin{aligned}\hat{w}(X') &= \int\hat{w}(X',X)\,dX \\ \hat{w} &= \int\hat{w}(X)\,dX\end{aligned}$$

the stakes are functions of the returns and risk coefficients. Solving the return equations will yield the formula for stakes.

A3.2 Banks investments risks

The weights of inverse uncertainty between banks for both loans and stakes[74] are given by:

$$(\bar{w}(X',X))^{-1}=1+\frac{1}{2}\left(\frac{\overline{IRG}(X',X)}{\widehat{IR^B}(X',X)}+\frac{\overline{IRG}(X',X)}{\xi^2}\right) \tag{166}$$

and those for banks investing investors are given by:

$$\left(\hat{w}_E^B(X',X)\right)^{-1}=1+2\frac{\widehat{IR^B}(X',X)}{\overline{IRG}(X',X)}+\frac{\widehat{IR^B}(X',X)}{\xi^2} \tag{167}$$

where:

$$w_E^B(X',X)=1-\bar{w}(X',X)-\hat{w}_E^B(X',X) \tag{168}$$

are the risk of investing in investors X' and banks X', respectively. These risks are depend on the level of investments these agents will themselves realize. These level are weighted by the level of uncertainty γ and $\bar{\gamma}$.

The factor $\overline{IRG}(X')$:

$$\overline{IRG}(X')=\left(\frac{\bar{\zeta}^2\overline{IR}(X')}{\left\langle\hat{w}_E^{(0)B}\left((X')',X'\right)\right\rangle_{(X')'}}+\frac{\xi^2}{\zeta^2}\frac{\overline{IR}(X')}{\widehat{IR^B}(X')}\right)$$

stands for the global risk of investing. It combines the relative risk to invest in a bank X' rather than in an investor X', $\frac{\overline{IR}(X')}{\widehat{IR^B}(X')}$, and the general propensity for banks to invest in X' compared to any other investment $\frac{\bar{\zeta}^2\zeta^2\overline{IR}(X')}{\left\langle\hat{w}_E^{(0)B}((X')',X')\right\rangle_{(X')'}}$.

[74] See section 5.2 for details.

In the above, the coefficient $\bar{\gamma}$ is the average uncertainty associated with distance-dependent investment paths for banks:

$$\bar{\gamma}^2 \simeq \left(\frac{1}{\bar{w}_1^{(0)}\left(\left(X'\right)', X'_{m-1}\right) ... \bar{w}_1^{(0)}\left(X'_1, X'\right)}\right)^{\frac{1}{m}}$$

the coefficients $\bar{w}_1^{(0)}\left(X'_1, X'\right)$ captures the local component of uncertainty perceived in neighboring banks, whereas ζ^2 and ξ^2 capture the global uncertainty in investors and firms returns respectively.Since they are considered constant across agents, they characterize the two types of agents' uncertainty

A3.3 Averages

A3.3.1 Average investors stakes

We denote z_0 the value of $\left\langle \hat{S}_E\left(X', X\right)\right\rangle$ and x, the value of $\left\langle \bar{S}_E\left(X', X\right)\right\rangle$ for $\left\langle R_{exc}\left(X\right)\right\rangle = 0$. Their values are functions of the level of uncertainty γ.

The average shares taken in, and loans granted by, investors in other investors are, respectively:

$$\begin{aligned} \left\langle \hat{S}_E\left(X', X\right)\right\rangle &= \left\langle \hat{S}_E\left(X'\right)\right\rangle = z_0\left(1 + \frac{1}{2}\frac{z_0^2}{D}\left\langle R_{exc}\left(X\right)\right\rangle\right) \qquad (169) \\ \left\langle \hat{S}_L\left(X', X\right)\right\rangle &= z_0\left(1 - \frac{1}{2}\left(1 - \frac{z_0^2}{D}\right)\left\langle R_{exc}\left(X\right)\right\rangle\right) \end{aligned}$$

where:

$$D = 1 - 5z_0 + 8z_0^2$$

and the total average stake is:

$$\begin{aligned} \left\langle \hat{S}\left(X', X\right)\right\rangle &= \left\langle \hat{S}_E\left(X', X\right)\right\rangle + \left\langle \hat{S}_L\left(X', X\right)\right\rangle \qquad (170) \\ &= 2z_0\left(1 - \frac{1}{4}\frac{\left(1 - 3z_0\right)\left(1 - 2z_0\right)}{D}\left\langle R_{exc}\left(X\right)\right\rangle\right) \end{aligned}$$

The stakes in firms are given by:

$$\begin{aligned} \left\langle S_E\left(X, X\right)\right\rangle &= \frac{1 - 2z_0}{2}\left(1 + \left(z_0\varepsilon\left(z_0\right) + \left(\frac{3}{4} - z_0\right)\right)\left\langle R_{exc}\left(X\right)\right\rangle\right) \qquad (171) \\ \left\langle S_L\left(X, X\right)\right\rangle &= \frac{1 - 2z_0}{2}\left(1 + \left(z_0 - \varepsilon\left(z_0\right)\left(\frac{1}{2} - z_0\right) - \frac{3}{8}\right)\left\langle R_{exc}\left(X\right)\right\rangle\right) \\ \left\langle S\left(X, X\right)\right\rangle &= \left\langle S_E\left(X, X\right)\right\rangle + \left\langle S_L\left(X, X\right)\right\rangle \\ &= \left(1 - 2z_0\right)\left(1 + \left(\frac{1}{2} + \varepsilon\left(z_0\right)\right)z_0\left\langle R_{exc}\left(X\right)\right\rangle\right) \end{aligned}$$

where:

$$\varepsilon\left(z_0\right) = \frac{z_0^2\left(1 - 4z_0\right)}{\left(1 - 5z_0 + 8z_0^2\right)}$$

Under excess return, firms seeking fresh capital will turn to equity creation rather than debt. Shares in firms $\left\langle S_E\left(X, X\right)\right\rangle$ increase, whereas loans $\left\langle S_L\left(X, X\right)\right\rangle$ decline. Nonetheless, total investment in firms $\left\langle S\left(X, X\right)\right\rangle$ increase. Besides, a relative increase in uncertainty will increase direct investment in firms.

A.3.3.2 Average banks stakes

For banks, the cross-shares in banks are given by:

$$\left\langle \bar{S}_E\left(X',X\right)\right\rangle = \left(1+\left(\frac{1-\left\langle \bar{S}_E\right\rangle_0}{\left(1-2\left\langle \bar{S}_E\right\rangle\right)}\left\langle S_E^{(e)}\right\rangle_0 - \left\langle \bar{S}_E^B\right\rangle_0 - \frac{\left\langle \hat{S}_E^B\right\rangle_0}{2}\right)\left\langle R_{exc}\left(X\right)\right\rangle\right)\left\langle \bar{S}_E\right\rangle_0 \tag{172}$$

and cross-loans by:

$$\left\langle \bar{S}_L\left(X',X\right)\right\rangle = \left(1+\left(\frac{\left\langle \bar{S}_E\right\rangle_0}{\left(1-2\left\langle \bar{S}_E\right\rangle\right)}\left\langle S_E^{(e)}\right\rangle_0 - \left\langle \bar{S}_E^B\right\rangle_0 - \frac{\left\langle \hat{S}_E^B\right\rangle_0}{2}\right)\left\langle R_{exc}\left(X\right)\right\rangle\right)\left\langle \bar{S}_E\right\rangle_0 \tag{173}$$

$$\begin{aligned}\left\langle \bar{S}\left(X',X\right)\right\rangle &= 2\left\langle \bar{S}_E\left(X',X\right)\right\rangle - \frac{\left\langle \bar{S}_E\left(X',X\right)\right\rangle}{\left(1-2\left\langle \bar{S}_E\left(X',X\right)\right\rangle\right)}\left\langle S_E^{(e)}\left(X,X\right)\right\rangle_0\left\langle R_{exc}\left(X\right)\right\rangle \\ &\simeq 2x-2x\left(\frac{\left\langle \hat{S}_E^B\right\rangle_0}{2}+\frac{\left\langle \bar{S}_E^B\right\rangle_0}{2\left(1-\left\langle \bar{S}_E\right\rangle_0\right)}\right)\left\langle R_{exc}\left(X\right)\right\rangle\end{aligned} \tag{174}$$

These results differ from the investors case. When firms' returns increase, banks gain to directly invest in firms or investors. This is explicit in the formula for banks shares in firms and investors:

$$\begin{aligned}\left\langle S_E^B\right\rangle &\simeq 1-6x+\frac{7}{4}x^2\left(1+5x\right)\left\langle R_{exc}\left(X\right)\right\rangle \\ &-\frac{1}{2}\left(1-x\right)^2 x\left\langle \hat{R}_{exc}\left(X\right)\right\rangle - \frac{1}{4}\left(1-x\right)x^2\left\langle \bar{R}_{exc}\left(X\right)\right\rangle\end{aligned} \tag{175}$$

and similarly:

$$\begin{aligned}\left\langle \hat{S}_E^B\left(X',X\right)\right\rangle &= 1-\left\langle S_E^B\right\rangle - \left\langle \bar{S}\left(X',X\right)\right\rangle \\ &\simeq 4x+2x\left(\frac{4-11x}{2}\right)\left\langle R_{exc}\left(X\right)\right\rangle \\ &+\frac{1}{2}\left(1-x\right)^2 x\left\langle \hat{R}_{exc}\left(X\right)\right\rangle + \frac{1}{4}\left(1-x\right)x^2\left\langle \bar{R}_{exc}\left(X\right)\right\rangle\end{aligned}$$

As expected, banks loans to investors and firms depend positively on the firms' returns $\left\langle f\left(X\right)\right\rangle$ and on the investors' shares z_0. They write:

$$\begin{aligned}&\left\langle \hat{S}_L^B\left(X'\right)\right\rangle \\ &= \kappa\left(1-2z_0\right)2z_0\left(1+z_0\left(1-\frac{\left(1-4z_0\right)\left(1-2z_0\right)}{2\left(1-5z_0+8z_0^2\right)}\right)\left\langle R_{exc}\left(X\right)\right\rangle \right. \\ &\left. +\left(1-2z_0\right)\left(\left\langle \hat{r}\left(X'\right)\right\rangle - \left\langle r\left(X\right)\right\rangle\right)\right)\end{aligned} \tag{176}$$

$$\begin{aligned}&\left\langle S_L^B\left(X,X\right)\right\rangle \\ &= \kappa\left(1-2z_0\right)^2 \\ &\times\left(1+z_0\left(1+\frac{\left(1-4z_0\right)z_0}{\left(1-5z_0+8z_0^2\right)}\right)\left\langle R_{exc}\left(X\right)\right\rangle + 2z_0\left(\left\langle r\left(X\right)\right\rangle - \left\langle \hat{r}\left(X'\right)\right\rangle_{\hat{w}_2}\right)\right)\end{aligned} \tag{177}$$

The increase of returns calls for more capital, both directly and indirectly, so that loans increase in proportion of the multiplier κ. The increase in loans is nonetheless bounded by the assumption of decreasing returns.

A3.3.3 Average disposable capital for investors and associated agregate stakes

Disposable capital for investors is:

$$\left\langle \hat{K} \right\rangle \left\| \hat{\Psi} \right\|^2 \simeq \frac{\hat{\mu} V \sigma_{\hat{K}}^2}{2} \left(\frac{\frac{\left\| \hat{\Psi}_0 \right\|^2}{\hat{\mu}} \left(1 - \hat{S}\right) + \tau \left\langle \hat{f} \right\rangle \hat{S}}{\left\langle \hat{f} \right\rangle \left(1 - \hat{S}\right)} \right)^2 \tag{178}$$

where $\left\langle \hat{f} \right\rangle$ is the average return for investors:

$$\left\langle \hat{f} \right\rangle \simeq \left\langle \hat{r}\left(X'\right) \right\rangle + \frac{f_a - f_b \left(\frac{C_0 + \bar{r}}{f_1(X)} \right)^{\frac{2}{r}}}{2} \tag{179}$$

and where f_a and f_b, τ are some coefficients[75], and the parameter $\left\| \hat{\Psi}_0 \right\|^2$ represents the average number of investors per sectors.

In the above,the parameter $\hat{\mu}$ represents the fluctuations in the number of agents within each sector. When $\hat{\mu}$ is high, the number of agents may fluctuate widely. In this case, the average is no longer a defining term for the level of disposable capital.

The associated average aggregate stakes are:

$$\begin{aligned} \left\langle \hat{S}_E\left(X'\right) \right\rangle &= \left\langle \hat{S}_E\left(X', X\right) \right\rangle \\ \left\langle \hat{S}_L\left(X'\right) \right\rangle &= \left\langle \hat{S}_L\left(X', X\right) \right\rangle \end{aligned}$$

$$\begin{aligned} \left\langle S_E\left(X\right) \right\rangle &= \left\langle \hat{S}_E\left(X, X\right) \right\rangle \frac{\left\langle \hat{K} \right\rangle \left\| \hat{\Psi} \right\|^2}{\left\langle K \right\rangle \left\| \Psi \right\|^2} \\ \left\langle S_L\left(X\right) \right\rangle &= \left\langle S_L\left(X, X\right) \right\rangle \frac{\left\langle \hat{K} \right\rangle \left\| \hat{\Psi} \right\|^2}{\left\langle K \right\rangle \left\| \Psi \right\|^2} \end{aligned}$$

A3.3.4 Disposable capital for banks and associated aggregate stakes

The disposable capital for banks $\left\langle \bar{K} \right\rangle \left\| \bar{\Psi} \right\|^2$ is given by:

$$\left\langle \bar{K} \right\rangle \left\| \bar{\Psi} \right\|^2 = 9 \frac{\sigma_{\hat{K}}^2 V \left\| \bar{\Psi}_0 \right\|^4 \left(1 - \bar{S}\right)^2}{2 \hat{\mu} \left\langle \bar{f} \right\rangle^2}$$

where the bank returns are given in first approximation by their returns of loans:

$$\left\langle \bar{f} \right\rangle \simeq (1 + \kappa) \bar{r}$$

A3.4 Sectoral returns and disposable capital for investors, banks and firms.

Having established the average solutions, we now refine the analysis to identify sector-specific equilibria. In contrast with the previous study, the incorporation of banks alters the distribution

[75] These coefficients are defined in Apppendix 5.4 in Gosselin and Lotz (2025).

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of investors' returns. The principle of a dual solution—whereby high and low return states coexist within the same sectors—remains valid for both banks and investors.

For investors, however, such a configuration arises only under specific conditions. When these conditions are not met, the system converges toward a unique solution characterized by returns clustered around the average across all investors. In what follows, we examine separately the return structures of investors and of banks.

A.3.4.1 Solutions of returns equations

A.3.4.1.1 Investors returns Investors' return per sector depend on three coefficients: the investors' estimation of risks in investors stakes, denoted z and defined by[76]:

$$z=\left\langle \hat{S}_E\left(X',X\right)\right\rangle$$

the relative strength of banks' investments in investors, denoted[77] x:

$$x=\left\langle \bar{S}_E\left(X',X\right)\right\rangle$$

and the relative average number of banks over that of investors, denoted P, defined by:

$$P=\frac{\left\|\bar{\Psi}_0\right\|^2}{\left\|\hat{\Psi}_0\right\|^2}$$

The higher the ratio P, the higher the importance of banking system and its impact on intermediation. The higher the uncertainty of banks about investors, the lower is x, and the lower is the impact of banks on investors. Under high uncertainty, banks reduce participations and loans. The higher the coefficient z, the higher will be the intermediation among investors, and consequently, the higher the impact of banks on investors.

Two types of solutions arise depending on these three parameters P and x and z.

First case: a unique solution We show that when P is above some threshold[78]:

$$P>Th$$

with:

$$Th=\frac{2\left(2z-1\right)^2\left(A-1\right)}{\left(1-2x\right)}+\frac{\sqrt{\left(2\left(2z-1\right)^2\left(A-1\right)\right)^2-\left(1-2x\right)\left(\left(A-1\right)\left(4-4z-\frac{1}{z}\right)\right)}}{\left(1-2x\right)}$$

and:

$$A=\frac{2-24z+112z^2-224z^3+160z^4}{1-9z+26z^2-12z^3+8z^4}$$

investors' return equations per sector (102) have only a unique solution, which is close to the average solution:

$$\hat{R}_{exc}\left(X\right)\simeq z\left(\left\langle \hat{R}_{exc}\left(X'\right)\right\rangle\right)+\left(1-z\right)R_{exc}\left(X\right) \tag{180}$$

[76]The link between z and z_0, the value of $\left\langle \hat{S}_E\left(X',X\right)\right\rangle$ under the constraint $\left\langle f\left(X\right)\right\rangle-\left\langle \bar{r}\left(X\right)\right\rangle=0$, is given in Appendix 5.4.1 of Gosselin and Lotz (2025). In first approximation, we can consider $z=z_0$.

[77]To the first approximation we identify $x=\left\langle \bar{S}_E\left(X',X\right)\right\rangle=\left\langle \bar{S}_E\right\rangle_0$ with $\left\langle \bar{S}_E\right\rangle_0$ defined in section 12.1.3.. First order corrections to this approximation are defined in Appendix 6.5.2.

[78]See Appendix 9.

Second case: two solutions On the contrary, when the relative average number of banks is below the threshold:

$$P < Th$$

investors' return equations per sector (102) have two solutions, as in Gosselin and Lotz (2025). These two solutions for investors' excess returns depend on firms' excess returns per sector and average returns in the system. They write[79]:

$$\hat{R}^{H}_{exc}(X) = \frac{\hat{R}^{H,0}_{exc}(X)}{2} + \sqrt{\left(\frac{\hat{R}^{H,0}_{exc}(X)}{2}\right)^2 - \frac{1-2z}{1-z}\frac{\langle \bar{r}(X)\rangle}{|a(z,P)|}\hat{R}^{L,0}_{exc}(X)} \tag{181}$$

and:

$$\hat{R}^{L}_{exc}(X) = \frac{\hat{R}^{H,0}_{exc}(X)}{2} - \sqrt{\left(\frac{\hat{R}^{H,0}_{exc}(X)}{2}\right)^2 - \frac{1-2z}{1-z}\frac{\langle \bar{r}(X)\rangle}{|a(z,P)|}\hat{R}^{L,0}_{exc}(X)} \tag{182}$$

with:

$$\hat{R}^{H,0}_{exc}(X) = \frac{\frac{1-2z}{1-z}\langle \bar{r}(X)\rangle + z\left(b(z,L)\left\langle \hat{R}_{exc}(X')\right\rangle + \frac{1}{2}\frac{1-2z}{(z-1)^2}R_{exc}(X)\right)}{2|a(z,P)|} \tag{183}$$

and:

$$\hat{R}^{L,0}_{exc}(X) = z\left\langle \hat{R}_{exc}(X')\right\rangle + \frac{(1-z)}{2}R_{exc}(X) \tag{184}$$

where the functions $a(z,P)$ and $b(z,L)$ are defined in Appendix 10.5. Note that the excess return of firms $R_{exc}(X)$ is given by $(f_1(X) - \langle \hat{r}(X')\rangle)$ under constant returns to scale, and by $\frac{1}{2}f_a - \frac{1}{2}f_b\left(\frac{C_0+\bar{r}}{f_1(X)}\right)^{\frac{1}{r}}$ under decreasing returns to scale[80]. The interpretations will be similar in both cases, but with dampened amplitudes for decreasing returns.

Obviously, the two solutions (181) and (182) differ whether we add or substract the second term:

$$\sqrt{\left(\frac{\hat{R}^{H,0}_{exc}(X)}{2}\right)^2 - \frac{1-2z}{1-z}\frac{\langle \bar{r}(X)\rangle}{|a(z,P)|}\hat{R}^{L,0}_{exc}(X)} \tag{185}$$

When it is added, it is the high-return solution (181), in which returns are relatively higher than otherwise. When it is substracted, it yields the low-return solution (182).

When excess returns are relatively small compared to the interest rate, these two solutions can be expressed as:

$$\hat{R}^{H}_{exc}(X) = \hat{R}^{H,0}_{exc}(X) - \hat{R}^{L,0}_{exc}(X) \tag{186}$$

and:

$$\hat{R}^{L}_{exc}(X) = \hat{R}^{L,0}_{exc}(X)\left(1 - \frac{2|a(z,P)|(1-z)}{(1-2z)}\left(\frac{\left\langle \hat{R}_{exc}(X)\right\rangle}{\langle \bar{r}\rangle} + \frac{R_{exc}(X)}{\langle \bar{r}\rangle}\right)\right) \tag{187}$$

Note that the high-return solution only occurs under some conditions. For instance, when uncertainty is high, returns $\hat{R}^{H}_{exc}(X)$ would have to be very high to compensate for high uncertainty, which rules out this solution practically. The high-return solution therefore only occurs when the risk perception of investors is low.

[79]See Appendix 7 in Gosselin and Lotz (2025) for the derivation.
[80]See Appendix 8.2.

A.3.4.1.2 Banks returns For banks, there are two solutions that have a similar form to those of investors:

$$\bar{R}_{exc}(X)=\frac{\bar{R}_{exc}^{H,0}(X)}{2}\pm\sqrt{\left(\left(\frac{\bar{R}_{exc}^{H,0}(X)}{2}\right)^2-(\kappa+2)(\kappa+1)\frac{1-x}{x}\bar{R}_{exc}^{L,0}(X)\right)^2}$$

with:

$$\bar{R}_{exc}^{H,0}(X)=(\kappa+2)(\kappa+1)\left(\frac{1-x}{2x}\langle\bar{r}(X')\rangle+\frac{(\langle f(X)\rangle-\langle\bar{r}(X')\rangle)}{4}+\left(\left\langle\hat{f}(X')\right\rangle-\langle\bar{r}(X')\rangle\right)x\right)$$

and:

$$\begin{aligned}&\bar{R}_{exc}^{L,0}(X)=\left\langle\bar{S}_E(X',X)\right\rangle_{X'}\left\langle\overline{DF}(X')\right\rangle\left(\left\langle\bar{f}(X')\right\rangle-\bar{r}\right)\\&+\left\langle\hat{S}_E^B(X',X)\right\rangle_{X'}\left\langle\widehat{DF}(X')\right\rangle\left(\left\langle\hat{f}(X')\right\rangle-\bar{r}\right)+S_E^B(X,X)\left(f_1(X)_{dr}-\bar{r}\right)\end{aligned}$$

Since $\frac{2x}{(\kappa+2)(\kappa+1)}<<1$, the low-return solution is close to the average:

$$\begin{aligned}&\bar{R}_{exc}^{L}(X)\\\simeq\;&\hat{R}_{exc}^{L,0}(X)\left(1-\frac{2x}{(1-x)(\kappa+2)(\kappa+1)}\bar{R}_{exc}^{H,0}(X)\right)\\\simeq\;&\bar{R}_{exc}^{L,0}(X)\end{aligned}$$

and this rewrites in terms of banks uncertainty x as:

$$\bar{R}_{exc}(X)\simeq x\left(\left\langle\bar{f}(X')\right\rangle-(\kappa+1)\bar{r}\right)+4x\left(\left\langle\hat{f}(X')\right\rangle-\bar{r}\right)+(1-6x)(f(X)-\bar{r})$$

As an investor, that is loans excluded, banks' returns combine their investments in other banks, in investors and in firms. The coefficients depend on uncertainty. For high uncertainty, banks mainly invest in firms, and diversify under low uncertainty. In this case the shares in investors are larger than in other banks, since the intermediation through investors is more direct than a path bank-investors-firms.

The high-return solution writes:

$$\bar{R}_{exc}^{H}(X)\simeq\bar{R}_{exc}^{H,0}(X)$$

The returns are written in terms of sectoral returns and their averages. The model is thus specified by the values of these quantities.

A.3.4.1.3 Firms returns Under decreasing returns to scale, firms' returns are:

$$f(X)=\frac{f_1(X)}{\left(K_X\left|\hat{\Psi}(X)\right|^2\right)^r}-\frac{C}{K_X}-C_0 \tag{188}$$

These returns depend on the sectoral disposable capital that will be obtained below.

A3.4.2 Disposable capital per sector

Under decreasing returns to scale, firms' returns are given by (188). For each solution (181), the disposable capital for banks is given by:

$$\bar{K}_X \left|\bar{\Psi}(X)\right|^2 \simeq \langle\bar{K}\rangle \|\bar{\Psi}\|^2 \left(\frac{\langle\bar{f}\rangle}{\left(\left(1-\langle\bar{S}\rangle\right)\bar{f}(X)+\left\langle\bar{S}(X',X)\right\rangle_{X'}\langle\bar{f}\rangle\right)}\bar{I}_{X/\langle X'\rangle}\right)^2 \tag{189}$$

where:

$$\bar{I}_{X/\langle X'\rangle} = \frac{\frac{\langle\bar{S}(X',X)\rangle_X}{1-\langle\bar{S}(X',X)\rangle_{X'}}}{\frac{\langle\bar{S}(X',X)\rangle}{1-\langle\bar{S}(X',X)\rangle}}$$

which measures the level of investment in investor X by other sectors, $\frac{\langle\bar{S}(X',X)\rangle_X}{1-\langle\bar{S}(X',X)\rangle_{X'}}$, with respect to the level of investment in the rest of the market, $\frac{\langle\bar{S}(X',X)\rangle}{1-\langle\bar{S}(X',X)\rangle}$. It is also weighted by the ratio:

$$\frac{\langle\bar{f}\rangle}{\left(\left(1-\langle\bar{S}\rangle\right)\bar{f}(X)+\left\langle\bar{S}(X',X)\right\rangle_{X'}\langle\bar{f}\rangle\right)}$$

that measures a saturation effect: a high level of returns limits the level of capital that produces this level of return. It is not solely the return $\bar{f}(X)$ that matters, but its weighted combination with average returns $\left\langle\bar{S}(X',X)\right\rangle_{X'}\langle\bar{f}\rangle$ which translates the share of capital reinvested in the market by banks X.

The capital levels per sector for investors are:

$$\hat{K}_X \left|\hat{\Psi}(X)\right|^2 \simeq \left\langle\hat{K}\right\rangle \left\|\hat{\Psi}\right\|^2 \left(\frac{\langle\hat{g}\rangle}{\hat{g}(X)} I_{X/\langle X'\rangle}\right)^2 \tag{190}$$

where $\langle\hat{g}\rangle$ and $\hat{g}(X)$ are given by:

$$\langle\hat{g}\rangle = \left\langle\hat{f}\right\rangle + \frac{\left(\hat{S}_E^B+\hat{S}_L^B\right)}{1-\bar{S}} \frac{\langle\bar{K}\rangle\|\bar{\Psi}\|^2}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\langle\bar{f}\rangle$$

$$\hat{g}(X) = \left(\hat{f}(X) + \frac{\left(\left\langle\hat{S}_E^B(X,X')\right\rangle_{X'} + \left\langle\hat{S}_L^B(X,X')\right\rangle_{X'}\right)\frac{\langle\bar{K}\rangle\|\bar{\Psi}\|^2}{\langle\hat{K}\rangle\|\hat{\Psi}\|^2}\langle\bar{f}\rangle}{1-\langle\bar{S}\rangle}\right) + \left\langle\hat{S}(X',X)\right\rangle_{X'}\langle\hat{g}\rangle$$

The average investors return are defined by (179) and investors X returns $\hat{f}(X)$ are given by:

$$\hat{f}(X) \simeq \hat{r}(X) + \frac{1}{2}f_a - \frac{1}{2}f_b\left(\frac{C_0+\hat{r}(X)}{f_1(X)}\right)^{\frac{1}{r}}$$

Disposable capital (190) is proportional to the ratio:

$$\hat{I}_{X/\langle X'\rangle} = \frac{\frac{\langle\hat{S}(X,X')\rangle_{X'}}{1-\langle\hat{S}(X,X')\rangle_{X'}}}{\frac{\langle\hat{S}(X,X')\rangle}{1-\langle\hat{S}(X,X')\rangle}} \tag{191}$$

which measures the level of investment in investor X by other sectors, $\frac{\left\langle \hat{S}(X,X')\right\rangle_{X'}}{1-\left\langle \hat{S}(X,X')\right\rangle_{X'}}$, with respect to the level of investment in the rest of the market, $\frac{\left\langle \hat{S}(X,X')\right\rangle}{1-\left\langle \hat{S}(X,X')\right\rangle}$.

It is also weighted by the ratio $\frac{\langle\hat{g}\rangle}{\hat{g}(X)}$ that measures a saturation effect: a high level of returns limits the level of capital that produces this level of return. It is not solely the return $\hat{f}(X)$ that matters, but its weighted combination with average returns $\left\langle \hat{S}(X',X)\right\rangle_{X'}\left\langle \hat{f}\right\rangle$ which translates the share of capital reinvested in the market by investors X.

The disposable capital for firms is given by:

$$K_X\left|\Psi(X)\right|^2\simeq\left(1-\left(S(X,X)\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{\left(K_X\left\|\Psi(X)\right\|^2\right)_0}+S^B(X,X)\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\left(K_X\left\|\Psi(X)\right\|^2\right)_0}\right)\right)\left(K_X\left\|\Psi(X)\right\|^2\right)_0 \tag{192}$$

where:

$$\left(K_X\left\|\Psi(X)\right\|^2\right)_0=\left(\left(\frac{2\epsilon}{3\sigma_{\hat{K}}^2}\right)^{\frac{r}{2}}\frac{f_1(X)}{C_0+\bar{r}}\right)^{\frac{2}{r}}$$

is the firms X disposable capital in the absence of investor participation, i.e. when $S(X,X)=0$.

Under both solutions, firms' capital increases with firms' productivity $f_1(X)$, and investors' X disposable capital increase with firms' productivity $f_1(X)$. This refines our results for the average capital found in step 1, since this increase in firms' productivity in any given sector X increases investments in investors X from other sectors. However, investors' disposable capital is higher under the high-return solution than under the low-return solution, since investors attract more capital so that, under this solution the ratio $I_{X/\langle X'\rangle}$ is particularly high.

A3.4.3 Capital ratio per sector

$$\frac{\langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}\simeq\left(\frac{\left(\left(1-\langle\bar{S}\rangle\right)\bar{f}(X)+\left\langle\bar{S}(X',X)\right\rangle_{X'}\langle\bar{f}\rangle\right)}{\langle\bar{f}\rangle}\bar{I}_{X/\langle X'\rangle}\right)^2 \tag{193}$$

and using (190), the ratio bank's average-investors' sector is:

$$\frac{\langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^2}{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}\simeq\left(\frac{\hat{g}(X)}{\langle\bar{g}\rangle}\right)^2\frac{\left\|\bar{\Psi}_0\right\|^4}{\left\|\hat{\Psi}_0\right\|^4\left(1+\frac{\left\|\bar{\Psi}_0\right\|^2}{\left\|\hat{\Psi}_0\right\|^2}\left\langle\hat{S}_L^B\right\rangle\right)} \tag{194}$$

The ratio banks disposable capital to firms disposabl capital:

$$\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\simeq\frac{18\sigma_{\hat{K}}^2V\left\|\bar{\Psi}_0(X)\right\|^4}{\left(\bar{f}(X)+\frac{\left\langle\bar{S}(X',X)\right\rangle_{X'}}{\left(1-\langle\bar{S}\rangle\right)}\langle\bar{f}\rangle\right)^2\hat{\mu}\left(\left(\frac{2\epsilon}{3\sigma_{\hat{K}}^2}\right)^{\frac{r}{2}}\frac{f_1(X)}{C_0+\frac{S_L(X)}{1-S_E(X)}\bar{r}}\right)^{\frac{2}{r}}} \tag{195}$$

and the ratio investors disposable capital to firms disposable capital:

$$\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\simeq\frac{18\sigma_{\hat{K}}^2V\left\|\hat{\Psi}_0(X)\right\|^4}{\hat{\mu}F_1^2\left(\left(\frac{2\epsilon}{3\sigma_{\hat{K}}^2}\right)^{\frac{r}{2}}\frac{f_1(X)}{C_0+\frac{S_L(X)}{1-S_E(X)}\bar{r}}\right)^{\frac{2}{r}}}$$

where F_1 is defined in Appendix 9.

A3.5 Formula for stakes

Once the sectoral returns, the sectoral levels of disposable capital and the uncertainty coefficients (166), (167), (168) are obtained, the cross-sectoral stakes are derived using the stakes equations given in Appendix 3.1. The formulas for these cross-sectoral stakes involve several type of relative returns for banks and investors. The detailed results are not necessary in the present context since we consider the stakes values as some given parameters from which the remaining fiber variables are defined. The precise formula are given in Gosselin and Lotz (2025b).

Appendix 4 Basis Stability

A4.1 Quadratic order dynamics for banks

We consider the dynamic fluctuations:

$$\begin{pmatrix} \delta\bar{S}\left(X',X,\theta\right) \\ \delta\hat{S}\left(X',X,\theta\right) \\ \delta S^{T}\left(X,,\theta-1\right) \end{pmatrix} = \begin{bmatrix} a & 0 & \frac{\bar{h}\left(X',X\right)}{2}c \\ \frac{\hat{h}\left(X',X\right)}{\bar{h}\left(X',X\right)}d & -1 & \frac{\hat{h}\left(X',X\right)}{2}f \\ \frac{g}{\frac{\bar{h}\left(X',X\right)}{2}} & 0 & i \end{bmatrix} \begin{pmatrix} \delta\bar{S}\left(X',X,\theta-1\right) \\ \delta\hat{S}\left(X',X,\theta-1\right) \\ \delta S^{T}\left(X,,\theta-2\right) \end{pmatrix}$$
$$+\int dX'' \begin{bmatrix} v\left(X,X'\right) & w\left(X,X'\right) & 0 \\ 0 & t\left(X,X'\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \delta\bar{S}\left(X'',X,\theta-1\right) \\ \delta\hat{S}\left(X'',X,\theta-1\right) \\ \delta S^{T}\left(X,,\theta-2\right) \end{pmatrix}$$
$$+\begin{pmatrix} \delta\bar{S}\left(X',X,\theta\right) \\ \delta\hat{S}\left(X',X,\theta\right) \\ \delta S^{T}\left(X,,\theta-1\right) \end{pmatrix}^{t} \begin{bmatrix} \bar{M} \\ \hat{M} \\ M^{T} \end{bmatrix} \begin{pmatrix} \delta\bar{S}\left(X',X,\theta\right) \\ \delta\hat{S}\left(X',X,\theta\right) \\ \delta S^{T}\left(X,,\theta-1\right) \end{pmatrix}$$

with:

$$\begin{aligned} a = & \left\{\left(\frac{1}{2}H\left(X\right)\bar{S}\left(X\right)-\frac{\bar{S}\left(X',\theta-1\right)\partial_{\bar{f}(X)}\left\langle\bar{K}_{X}\right\rangle\left|\bar{\Psi}\left(X\right)\right|^{2}}{\left\langle\bar{K}_{X'}\right\rangle\left|\bar{\Psi}\left(X'\right)\right|^{2}}\right)\frac{\left(\bar{f}\left(X\right)-\bar{r}\right)}{1-\left(\bar{S}_{1}\left(X',\theta-1\right)\right)}\right. \\ & -\left(H\left(X\right)\bar{S}_{1}\left(X\right)-\frac{\bar{S}_{1}\left(X',\theta-1\right)\partial_{\bar{f}(X)}\left\langle\bar{K}_{X}\right\rangle\left|\bar{\Psi}\left(X\right)\right|^{2}}{\left\langle\bar{K}_{X'}\right\rangle\left|\bar{\Psi}\left(X'\right)\right|^{2}}\right)\frac{\left(1-\left(\bar{S}\left(X,\theta-1\right)\right)\right)\left(\bar{f}\left(X\right)-\bar{r}\right)}{\left(1-\left(\bar{S}_{1}\left(X',\theta-1\right)\right)\right)^{2}} \\ & \left.+\left(S_{1}^{B}\left(X,\theta-1\right)+S_{2}^{B}\left(X,\theta-1\right)\right)\frac{\partial_{\bar{f}(X)}\bar{K}_{X}\left|\bar{\Psi}\left(X\right)\right|^{2}}{\bar{K}_{X}\left|\bar{\Psi}\left(X\right)\right|^{2}}S_{1}^{B}\left(X,X\right)\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-1\right)}\right\} \end{aligned}$$

$$
\begin{aligned}
c &= \Bigg\{h_1^B(X)\left(\langle\bar{h}(X)\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)(f(X)-\bar{r})\frac{\partial f(X)}{\partial S^T(X,\theta-2)}-\frac{\langle\bar{h}(X',X)\rangle}{2}h_1^B(X)\frac{1-\langle\bar{S}(X')\rangle}{1-\langle\bar{S}_1(X')\rangle}\left(\langle\bar{f}(X')\rangle-\bar{r}\right)\\
&-\left\langle\hat{h}_1^B(X',X)\right\rangle\left\langle h_1^B(X)\right\rangle\frac{1-\left\langle\hat{S}(X')\right\rangle+\left\langle\hat{S}_1^B(X')\right\rangle+\left\langle\hat{S}_2^B(X')\right\rangle}{1-\left\langle\hat{S}_1(X')\right\rangle+\left\langle\hat{S}_1^B(X')\right\rangle}\left(\left\langle\hat{f}(X')\right\rangle-\bar{r}\right)\\
&+S_1^B(X,X)\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\Bigg\{h_1^B(X)\left(\langle\bar{h}(X)\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)\\
&-\left(S_1^B(X,\theta-1)+S_2^B(X,\theta-1)\right)\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\\
&+\left(\frac{\frac{\hat{h}(X)}{2}S(X)}{1+\left(\hat{h}(X)\left(\frac{f(X)+\bar{r}(X)}{2}-\frac{\left\langle\hat{f}(X')\right\rangle_{\hat{h}_1}+\langle\hat{r}(X')\rangle_{\hat{h}_2}}{2}\right)\right)}+\frac{\partial_{f(X)}\left(\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}\right)\Bigg\}\Bigg\}\frac{\partial f(X)}{\partial S^T(X,\theta-2)}
\end{aligned}
$$

$$
d=S_1(X,X,\theta-1)\left(S_1^B(X,\theta-1)+S_2^B(X,\theta-1)\right)\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}<<1
$$

$$
\begin{aligned}
f=\Bigg\{&-\frac{\left(1-\left\langle\hat{S}(X')\right\rangle\right)\left(\left\langle\hat{f}(X')\right\rangle-\bar{r}\right)}{1-\left\langle\hat{S}_1(X')\right\rangle}\frac{\left\langle\hat{h}(X',X)\right\rangle\langle h(X)\rangle}{4}\\
&+\frac{\frac{h(X)}{4}\left(1+\hat{h}(X)\right)S_1(X)}{1+\left(\hat{h}(X)\left(f(X)-\frac{\left\langle\hat{f}(X')\right\rangle_{\hat{h}_1}+\langle\hat{r}(X')\rangle_{\hat{h}_2}}{2}\right)+\frac{h(X)}{2}\left(f(X)-\bar{r}(X)\right)\right)}(f(X)-\bar{r})\\
&+S_1(X,X,\theta-1)\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\\
&\times\left(h_1^B(X)\left(\langle\bar{h}(X)\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)-\left(S_1^B(X,\theta-1)+S_2^B(X,\theta-1)\right)\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\right.\\
&\left.+\frac{\frac{\hat{h}(X)}{2}S(X)}{1+\left(\hat{h}(X)\left(\frac{f(X)+\bar{r}(X)}{2}-\frac{\left\langle\hat{f}(X')\right\rangle_{\hat{h}_1}+\langle\hat{r}(X')\rangle_{\hat{h}_2}}{2}\right)\right)}+\frac{\partial_{f(X)}\left(\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}\right)\Bigg\}\\
&\times\frac{\partial f(X)}{\partial S^T(X,\theta-2)}
\end{aligned}
$$

$$
\begin{aligned}
g &= \left(S_1^B(X,\theta-1)+S_2^B(X,\theta-1)\right)\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}\\
i &= \Bigg\{h_1^B(X)\left(\langle\bar{h}(X)\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)-\left(S_1^B(X,\theta-1)+S_2^B(X,\theta-1)\right)\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\\
&+\left(\frac{\frac{\hat{h}(X)}{2}S(X)}{1+\left(\hat{h}(X)\left(\frac{f(X)+\bar{r}(X)}{2}-\frac{\left\langle\hat{f}(X')\right\rangle_{\hat{h}_1}+\langle\hat{r}(X')\rangle_{\hat{h}_2}}{2}\right)\right)}+\frac{\partial_{f(X)}\left(\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}\right)\Bigg\}\frac{\partial f(X)}{\partial S^T(X,\theta-2)}
\end{aligned}
$$

and the non local interactions:

$$v(X,X') = \frac{\bar{h}(X'',X')}{\bar{h}(X',X)}\bar{S}_1(X',X)\frac{1-\langle\bar{S}(X')\rangle}{1-\langle\bar{S}_1(X')\rangle}$$

$$w(X,X') = \frac{\bar{h}(X'',X')}{\hat{h}(X',X)}\hat{S}_1^B(X',X)\frac{1-\left\langle\hat{S}(X')\right\rangle+\left\langle\hat{S}_1^B(X')\right\rangle+\left\langle\hat{S}_2^B(X')\right\rangle}{1-\left\langle\hat{S}_1(X')\right\rangle+\left\langle\hat{S}_1^B(X')\right\rangle}$$

$$t(X,X') = \frac{\hat{h}(X'',X')}{\hat{h}(X',X)}\hat{S}_1(X',X)\frac{1-\left\langle\hat{S}(X')\right\rangle}{1-\left\langle\hat{S}_1(X')\right\rangle}$$

The quadratic corrections are given by the matrices:

$$\bar{M} = \begin{bmatrix} \frac{\delta}{\delta\bar{f}(X,\theta-1)}a & 0 & \frac{\bar{h}(X',X)}{2}\frac{\delta}{\delta\bar{f}(X,\theta-1)}c \\ \frac{\delta}{\delta\hat{f}(X,\theta-1)}a & 0 & \frac{\bar{h}(X',X)}{2}\frac{\delta}{\delta\hat{f}(X,\theta-1)}c \\ \frac{\delta}{\delta S^T(X,,\theta-2)}a & 0 & \frac{\bar{h}(X',X)}{2}\frac{\delta}{\delta S^T(X,,\theta-2)}c \end{bmatrix}, \hat{M} = \begin{bmatrix} \frac{\hat{h}(X',X)}{\bar{h}(X',X)}\frac{\delta}{\delta\bar{f}(X,\theta-1)}d & 0 & \frac{\hat{h}(X',X)}{2}\frac{\delta}{\delta\bar{f}(X,\theta-1)}f \\ \frac{\hat{h}(X',X)}{\bar{h}(X',X)}\frac{\delta}{\delta\hat{f}(X,\theta-1)}d & 0 & \frac{\hat{h}(X',X)}{2}\frac{\delta}{\delta\hat{f}(X,\theta-1)}f \\ \frac{\hat{h}(X',X)}{\bar{h}(X',X)}\frac{\delta}{\delta S^T(X,,\theta-2)}d & 0 & \frac{\hat{h}(X',X)}{2}\frac{\delta}{\delta S^T(X,,\theta-2)}f \end{bmatrix}$$

$$M^T = \begin{bmatrix} \frac{2}{h(X',X)}\frac{\delta}{\delta\bar{f}(X,\theta-1)}g & 0 & \frac{\delta}{\delta\bar{f}(X,\theta-1)}i \\ \frac{2}{h(X',X)}\frac{\delta}{\delta\hat{f}(X,\theta-1)}g & 0 & \frac{\delta}{\delta\hat{f}(X,\theta-1)}i \\ \frac{2}{h(X',X)}\frac{\delta}{\delta S^T(X,,\theta-2)}g & 0 & \frac{\delta}{\delta S^T(X,,\theta-2)}i \end{bmatrix}$$

We assume $\left(\bar{f}(X)-\bar{r}\right) << 1$ and $\left(\hat{f}(X)-\bar{r}\right) << 1$ due to decreasing returns and large amounts of loans. We will use:

$$\delta S_1(X,X,\theta-1) = \left(\frac{\frac{\hat{h}(X)}{2}S_1(X,X)}{1+\left(\hat{h}(X)\left(\frac{f(X)+\bar{r}(X)}{2}-\frac{\left\langle\hat{f}(X')\right\rangle_{\hat{h}_1}+\langle\hat{r}(X')\rangle_{\hat{h}_2}}{2}\right)\right)}\right)\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\delta S^T(X,\theta-2)$$

writtn as:

$$\begin{aligned}\delta S_1(X,\theta-1) &= \left(\frac{\frac{\hat{h}(X)}{2}S_1(X)}{1+\left(\hat{h}(X)\left(\frac{f(X)+\bar{r}(X)}{2}-\frac{\left\langle\hat{f}(X')\right\rangle_{\hat{h}_1}+\langle\hat{r}(X')\rangle_{\hat{h}_2}}{2}\right)\right)}+\frac{\partial_{f(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S_1(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\right) \\ &\times\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\delta S^T(X,\theta-2) \\ &+\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S_1(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\delta\hat{f}(X,\theta-1)\end{aligned}$$

$$\begin{aligned}
&\delta S\left(X,\theta-1\right)\\
=&\left(\frac{\frac{\hat{h}(X)}{2}S\left(X\right)}{1+\left(\hat{h}\left(X\right)\left(\frac{f(X)+\bar{r}(X)}{2}-\frac{\left\langle\hat{f}(X')\right\rangle_{\hat{h}_1}+\left\langle\hat{r}(X')\right\rangle_{\hat{h}_2}}{2}\right)\right)}+\frac{\partial_{f(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\right)S\left(X\right)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}}\right)\\
&\times\frac{\partial f\left(X\right)}{\partial S^T\left(X,\theta-2\right)}\delta S^T\left(X,\theta-2\right)\\
&+\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\right)S\left(X\right)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}}\delta\hat{f}\left(X,\theta-1\right)
\end{aligned}$$

$$\begin{aligned}
&\delta S_1^B\left(X,X,\theta-1\right)\\
=&\;h_1^B\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_1^B\left(X\right)\right\rangle\right)\frac{\partial f\left(X\right)}{\partial S^T\left(X,\theta-2\right)}\delta S^T\left(X,\theta-2\right)
\end{aligned}$$

$$\begin{aligned}
&\delta S_1^B\left(X,\theta-1\right)\\
=&\left(h_1^B\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_1^B\left(X\right)\right\rangle\right)-S_1^B\left(X,\theta-1\right)\frac{\partial_{f(X)}K_X\left|\Psi\left(X\right)\right|^2}{K_X\left|\Psi\left(X\right)\right|^2}\right)\frac{\partial f\left(X\right)}{\partial S^T\left(X,\theta-2\right)}\delta S^T\left(X,\theta-2\right)\\
&+S_1^B\left(X,\theta-1\right)\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}\left(X\right)\right|^2}{\bar{K}_X\left|\bar{\Psi}\left(X\right)\right|^2}\delta\bar{f}\left(X,\theta-1\right)
\end{aligned}$$

$$\begin{aligned}
\delta S_2^B\left(X,\theta-1\right)=&\;S_2^B\left(X,\theta-1\right)\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}\left(X\right)\right|^2}{\bar{K}_X\left|\bar{\Psi}\left(X\right)\right|^2}\delta\bar{f}\left(X,\theta-1\right)\\
&-S_2^B\left(X,\theta-1\right)\frac{\partial_{f(X)}K_X\left|\Psi\left(X\right)\right|^2}{K_X\left|\Psi\left(X\right)\right|^2}\frac{\partial f\left(X\right)}{\partial S^T\left(X,\theta-2\right)}\delta S^T\left(X,\theta-2\right)
\end{aligned}$$

Moreover:

$$\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}\left(X\right)\right|^2}{\bar{K}_X\left|\bar{\Psi}\left(X\right)\right|^2}\simeq-\frac{2}{\bar{f}\left(X\right)\left(1-\left\langle\bar{S}\right\rangle\right)+\left\langle\bar{S}\left(X',X\right)\right\rangle_{X'}\left\langle\bar{f}\right\rangle}$$

and:

$$\partial_{\bar{f}(X)}\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}\left(X\right)\right|^2}{\bar{K}_X\left|\bar{\Psi}\left(X\right)\right|^2}<<1$$

for $\kappa>>1$. Ultimately we will need the formula for $\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}$ and: $\frac{\partial_{f(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\right)S_1(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}}$. We have:

$$K_X\left|\Psi\left(X\right)\right|^2\simeq\left(\left(\frac{2\epsilon}{3\sigma_{\hat{K}}^2}\right)^{\frac{r}{2}}\frac{f_1\left(X\right)}{C_0+\frac{S_2(X)}{1-S_1(X)}\bar{r}}\right)^{\frac{2}{r}}$$

which implies:

$$\begin{aligned}\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2} &= \left(\partial_{f(X)}f_1(X)\right)\frac{\partial_{f_1(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\\ &= \frac{2}{r}\left(\partial_{f(X)}f_1(X)\right)\end{aligned}$$

For decreasing returns:

$$f_1(X)=K_X^r f^{dr}(X)+\frac{C}{K_X^{1-r}}+C_0K_X^r$$

$$\partial_{f(X)}f_1(X)=\frac{K_X^r}{f_1(X)}=\frac{1}{f^{dr}(X)+\frac{C}{K_X}+C_0}\simeq\frac{1}{\left(f^{dr}(X)+C_0\right)}$$

and:

$$\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\simeq\frac{2}{r}\frac{1}{f(X)+C_0}$$

Moreover, we use:

$$\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\simeq\frac{18\sigma_{\hat{K}}^2V\left\|\bar{\Psi}_0(X)\right\|^4}{\hat{\mu}F_1^2\left(\left(\frac{2\epsilon}{3\sigma_{\hat{K}}^2}\right)^{\frac{r}{2}}\frac{f_1(X)}{C_0+\frac{S_2(X)}{1-S_1(X)}\bar{r}}\right)^{\frac{2}{r}}}$$

and thus:

$$\frac{\partial_{f(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\right)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}}\simeq-\frac{2}{r}\frac{1}{f(X)+C_0}$$

$$\frac{\delta}{\delta S^T(X,,\theta-2)}\frac{\partial_{f(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\right)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}}\simeq-\frac{1}{f(X)+C_0}\frac{\partial_{f(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\right)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}}\frac{\delta f(X)}{\delta S^T(X,,\theta-2)}$$

$$\frac{\delta}{\delta S^T(X,,\theta-2)}\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\simeq-\frac{1}{f(X)+C_0}\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\frac{\delta f(X)}{\delta S^T(X,,\theta-2)}$$

$$\begin{aligned}&\delta\left(\left(S_1^B(X,\theta-1)+S_2^B(X,\theta-1)\right)\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\right)\\ =\;&\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)-\left(S_2^B(X,\theta-1)+S_1^B(X,\theta-1)\right)\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\left(1+\frac{\frac{1}{f(X)+C_0}}{\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}}\right)\right)\\ &\times\left(\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\right)\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\delta S^T(X,\theta-2)\\ &+\left(S_1^B(X,\theta-1)+S_2^B(X,\theta-1)\right)\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}\delta\bar{f}(X,\theta-1)\end{aligned}$$

$$
\begin{aligned}
&\frac{\partial}{\partial S^{T}(X,\theta-2)}\left(S_{1}^{B}(X,X)\left\{h_{1}^{B}(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_{1}^{B}(X)\right\rangle\right)\right.\right.\\
&\left.\left.-\left(S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)\right)\frac{\partial_{f(X)}K_{X}\left|\Psi(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}\right\}\right)\\
=\;&h_{1}^{B}(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_{1}^{B}(X)\right\rangle\right)\\
&\times\frac{\partial f(X)}{\partial S^{T}(X,\theta-1)}\left(h_{1}^{B}(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_{1}^{B}(X)\right\rangle\right)-\left(S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)\right)\frac{\partial_{f(X)}K_{X}\left|\Psi(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}\right)\\
&-S_{1}^{B}(X,X)\frac{\partial f(X)}{\partial S^{T}(X,\theta-1)}\left(h_{1}^{B}(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_{1}^{B}(X)\right\rangle\right)\right.\\
&\left.-\left(S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)\right)\left(1+\frac{1}{f(X)+C_{0}}\right)\frac{\partial_{f(X)}K_{X}\left|\Psi(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\delta}{\delta S^{T}(X,,\theta-2)}\left(S_{1}^{B}(X,X)\left(S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)\right)\right)\\
=\;&S_{1}^{B}(X,X)\left(h_{1}^{B}(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_{1}^{B}(X)\right\rangle\right)-\left(S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)\right)\frac{\partial_{f(X)}K_{X}\left|\Psi(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}\right)\\
&\times\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}+\left(S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)\right)h_{1}^{B}(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_{1}^{B}(X)\right\rangle\right)\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}
\end{aligned}
$$

The derivatives are obtained as:

$$
\begin{aligned}
&\frac{\delta}{\delta\bar{f}(X,\theta-1)}a\\
\simeq\;&\left\{\left(\frac{1}{2}H(X)\bar{S}(X)-\frac{\bar{S}(X',\theta-1)\,\partial_{\bar{f}(X)}\left\langle\bar{K}_{X}\right\rangle\left|\bar{\Psi}(X)\right|^{2}}{\left\langle\bar{K}_{X'}\right\rangle\left|\bar{\Psi}(X')\right|^{2}}\right)\frac{1}{1-\left(\bar{S}_{1}(X',\theta-1)\right)}\right.\\
&-\left(H(X)\bar{S}_{1}(X)-\frac{\bar{S}_{1}(X',\theta-1)\,\partial_{\bar{f}(X)}\left\langle\bar{K}_{X}\right\rangle\left|\bar{\Psi}(X)\right|^{2}}{\left\langle\bar{K}_{X'}\right\rangle\left|\bar{\Psi}(X')\right|^{2}}\right)\frac{\left(1-\left(\bar{S}(X,\theta-1)\right)\right)}{\left(1-\left(\bar{S}_{1}(X',\theta-1)\right)\right)^{2}}\\
&+\left(S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)\right)\\
&\left.\times\left(\frac{\partial_{\bar{f}(X)}\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}{\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}S_{1}^{B}(X,X)+h_{1}^{B}(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_{1}^{B}(X)\right\rangle\right)\right)\frac{\partial_{\bar{f}(X)}\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}{\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}\frac{\partial f(X)}{\partial S^{T}(X,\theta-1)}\right\}
\end{aligned}
$$

$$
\frac{\delta}{\delta\hat{f}(X,\theta-1)}a\simeq 0
$$

$$
\begin{aligned}
&\frac{\delta}{\delta S^{T}(X,\theta-2)}\left(S_{1}^{B}(X,X)\left(S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)\right)\right)\\
=\;&S_{1}^{B}(X,X)\left(h_{1}^{B}(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_{1}^{B}(X)\right\rangle\right)-\left(S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)\right)\frac{\partial_{f(X)}K_{X}\left|\Psi(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}\right)\\
&\times\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}+\left(S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)\right)h_{1}^{B}(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_{1}^{B}(X)\right\rangle\right)\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\delta}{\delta S^{T}\left(X,\theta-2\right)}a \\
\simeq\quad &\left(S_{1}^{B}\left(X,X\right)\left(h_{1}^{B}\left(X\right)\left(\left\langle \bar{h}\left(X\right)\right\rangle +\left\langle \hat{h}_{1}^{B}\left(X\right)\right\rangle \right)-\left(S_{1}^{B}\left(X,\theta-1\right)+S_{2}^{B}\left(X,\theta-1\right)\right)\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}\right)\right. \\
&\left.+\left(S_{1}^{B}\left(X,\theta-1\right)+S_{2}^{B}\left(X,\theta-1\right)\right)h_{1}^{B}\left(X\right)\left(\left\langle \bar{h}\left(X\right)\right\rangle +\left\langle \hat{h}_{1}^{B}\left(X\right)\right\rangle \right)\right)\left(\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-2\right)}\right)^{2}\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\delta}{\delta\bar{f}\left(X,\theta-1\right)}c \quad = \quad &\left\{h_{1}^{B}\left(X\right)\left(\left\langle \bar{h}\left(X\right)\right\rangle +\left\langle \hat{h}_{1}^{B}\left(X\right)\right\rangle \right)\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-2\right)}-\frac{\left\langle \bar{h}\left(X',X\right)\right\rangle }{2}h_{1}^{B}\left(X\right)\frac{1-\left\langle \bar{S}\left(X'\right)\right\rangle }{1-\left\langle \bar{S}_{1}\left(X'\right)\right\rangle }\right\} \\
&\times\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-2\right)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\delta}{\delta\hat{f}\left(X,\theta-1\right)}c \quad\simeq\quad &-\left\langle \hat{h}_{1}^{B}\left(X',X\right)\right\rangle \left\langle h_{1}^{B}\left(X\right)\right\rangle \frac{1-\left\langle \hat{S}\left(X'\right)\right\rangle +\left\langle \hat{S}_{1}^{B}\left(X'\right)\right\rangle +\left\langle \hat{S}_{2}^{B}\left(X'\right)\right\rangle }{1-\left\langle \hat{S}_{1}\left(X'\right)\right\rangle +\left\langle \hat{S}_{1}^{B}\left(X'\right)\right\rangle } \\
&+ZS_{1}^{B}\left(X,X\right)\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_{X}\left|\hat{\Psi}(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}\right)S\left(X\right)}{\frac{\hat{K}_{X}\left|\hat{\Psi}(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}}\left(\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-2\right)}\right)^{2}
\end{aligned}
$$

with:

$$
Z=\left(\frac{\frac{\hat{h}(X)}{2}}{1+\left(\hat{h}\left(X\right)\left(\frac{f(X)+\bar{r}(X)}{2}-\frac{\left\langle \hat{f}(X')\right\rangle _{\hat{h}_{1}}+\left\langle \hat{r}(X')\right\rangle _{\hat{h}_{2}}}{2}\right)\right)}+\frac{\partial_{f(X)}\left(\frac{\hat{K}_{X}\left|\hat{\Psi}(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}\right)}{\frac{\hat{K}_{X}\left|\hat{\Psi}(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}}\right)
$$

$$
\begin{aligned}
&\frac{\delta}{\delta S^{T}\left(X,,\theta-2\right)}c \\
\simeq\ &\left(\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-1\right)}\right)^{3} \\
&\times\left\{h_{1}^{B}\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_{1}^{B}\left(X\right)\right\rangle\right)\left(h_{1}^{B}\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_{1}^{B}\left(X\right)\right\rangle\right)\right.\right. \\
&\left.-\left(S_{1}^{B}\left(X,\theta-1\right)+S_{2}^{B}\left(X,\theta-1\right)\right)\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}+Z\right) \\
&+S_{1}^{B}\left(X,X\right)\left(-\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}\left(h_{1}^{B}\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_{1}^{B}\left(X\right)\right\rangle\right)\right)\right. \\
&-\left(S_{1}^{B}\left(X,\theta-1\right)+S_{2}^{B}\left(X,\theta-1\right)\right)\left(1+\frac{\frac{1}{f(X)+C_{0}}}{\frac{\partial_{f(X)}K_{X}\left|\Psi(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}}\right)\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}\Bigg) \\
&\left.\left.+\ Z^{2}S\left(X\right)-\frac{S\left(X\right)\partial_{f(X)}\left(\frac{\hat{K}_{X}\left|\hat{\Psi}(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}\right)}{\left(f\left(X\right)+C_{0}\right)\frac{\hat{K}_{X}\left|\hat{\Psi}(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}}\right)\right\} \\
&+c\frac{\frac{\partial^{2}f(X)}{\partial\left(S^{T}(X,\theta-2)\right)^{2}}}{\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}}+S_{1}^{B}\left(X,X\right)\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-1\right)}\left\{h_{1}^{B}\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_{1}^{B}\left(X\right)\right\rangle\right)\right. \\
&\left.-\left(S_{1}^{B}\left(X,\theta-1\right)+S_{2}^{B}\left(X,\theta-1\right)\right)\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}+Z\right\}\frac{\partial^{2}f\left(X\right)}{\partial\left(S^{T}\left(X,\theta-2\right)\right)^{2}}
\end{aligned}
$$

with:

$$
\begin{aligned}
&c\frac{\frac{\partial^{2}f(X)}{\partial\left(S^{T}(X,\theta-2)\right)^{2}}}{\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}} \\
\simeq\ &S_{1}^{B}\left(X,X\right)\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-1\right)}\left\{h_{1}^{B}\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_{1}^{B}\left(X\right)\right\rangle\right)\right. \\
&\left.-\left(S_{1}^{B}\left(X,\theta-1\right)+S_{2}^{B}\left(X,\theta-1\right)\right)\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}+Z\right\}\frac{\partial^{2}f\left(X\right)}{\partial\left(S^{T}\left(X,\theta-2\right)\right)^{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\delta}{\delta\bar{f}\left(X,\theta-1\right)}d \\
\simeq\ &\frac{\partial_{\bar{f}(X)}\bar{K}_{X}\left|\bar{\Psi}\left(X\right)\right|^{2}}{\bar{K}_{X}\left|\bar{\Psi}\left(X\right)\right|^{2}}d \\
=\ &S_{1}\left(X,X,\theta-1\right)\left(S_{1}^{B}\left(X,\theta-1\right)+S_{2}^{B}\left(X,\theta-1\right)\right)\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-1\right)}\left(\frac{\partial_{\bar{f}(X)}\bar{K}_{X}\left|\bar{\Psi}\left(X\right)\right|^{2}}{\bar{K}_{X}\left|\bar{\Psi}\left(X\right)\right|^{2}}\right)^{2}<<1
\end{aligned}
$$

so that we set:

$$
\frac{\delta}{\delta\hat{f}\left(X,\theta-1\right)}d\simeq 0
$$

$$\begin{aligned}
&\frac{\delta}{\delta S^{T}(X,,\theta-2)}d \\
\simeq\;& \left(\frac{\frac{\hat{h}(X)}{2}}{1+\left(\hat{h}(X)\left(\frac{f(X)+\bar{r}(X)}{2}-\frac{\left\langle \hat{f}(X')\right\rangle_{\hat{h}_1}+\left\langle \hat{r}(X')\right\rangle_{\hat{h}_2}}{2}\right)\right)}\right)\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}d \\
&+\left(h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle+\left\langle \hat{h}_1^B(X)\right\rangle\right)-\left(S_1^B(X,\theta-1)+S_2^B(X,\theta-1)\right)\frac{\left(1-\frac{1}{f(X)+C_0}\right)\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\right) \\
&\times\left(\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}\right)^2+\frac{\frac{\partial^2 f(X)}{\partial\left(S^{T}(X,\theta-2)\right)^2}}{\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}}d \\
\to\;& \left(\frac{\frac{\hat{h}(X)}{2}}{1+\left(\hat{h}(X)\left(\frac{f(X)+\bar{r}(X)}{2}-\frac{\left\langle \hat{f}(X')\right\rangle_{\hat{h}_1}+\left\langle \hat{r}(X')\right\rangle_{\hat{h}_2}}{2}\right)\right)}+\frac{\left(1-\frac{1}{f(X)+C_0}\right)h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle+\left\langle \hat{h}_1^B(X)\right\rangle\right)}{S_1^B(X,\theta-1)+S_2^B(X,\theta-1)}\right) \\
&\times\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}d+\frac{\frac{\partial^2 f(X)}{\partial\left(S^{T}(X,\theta-2)\right)^2}}{\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}}d
\end{aligned}$$

$$\begin{aligned}
\frac{\delta}{\delta \bar{f}(X,\theta-1)}f \;\simeq\;& -S_1(X,X,\theta-1)\left(S_1^B(X,\theta-1)+S_2^B(X,\theta-1)\right) \\
&\times\left[\frac{\partial f(X)}{\partial S^{T}(X,\theta-1)}\right]^2\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}
\end{aligned}$$

$$\begin{aligned}
&\frac{\delta}{\delta \hat{f}(X,\theta-1)}f \\
\simeq\;& -\frac{\left(1-\left\langle \hat{S}(X')\right\rangle\right)}{1-\left\langle \hat{S}_1(X')\right\rangle}\frac{\left\langle \hat{h}(X',X)\right\rangle\left\langle h(X)\right\rangle}{4}\frac{\partial f(X)}{\partial S^{T}(X,\theta-1)} \\
&+S_1(X,X,\theta-1)\left[\frac{\partial f(X)}{\partial S^{T}(X,\theta-1)}\right]^2 \\
&\times\left(h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle+\left\langle \hat{h}_1^B(X)\right\rangle\right)-\left(S_1^B(X,\theta-1)+S_2^B(X,\theta-1)\right)\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}+2Z\right) \\
&\times\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\right)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}}
\end{aligned}$$

$$
\begin{aligned}
&\frac{\delta}{\delta S^{T}(X,,\theta-2)}f \\
\simeq\;& \frac{\frac{h(X)}{4}\left(1+\hat{h}(X)\right)S_{1}(X)}{1+\left(\hat{h}(X)\left(f(X)-\frac{\left\langle\hat{f}(X')\right\rangle_{\hat{h}_{1}}+\left\langle\hat{r}(X')\right\rangle_{\hat{h}_{2}}}{2}\right)+\frac{h(X)}{2}\left(f(X)-\bar{r}(X)\right)\right)}\left(\frac{\partial f(X)}{\partial S^{T}(X,\theta-1)}\right)^{2}\frac{\partial_{f(X)}K_{X}\left|\Psi(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}} \\
&+S_{1}(X,X,\theta-1)\left(\frac{\partial f(X)}{\partial S^{T}(X,\theta-1)}\right)^{3}\frac{\frac{\hat{h}(X)}{2}}{1+\left(\hat{h}(X)\left(\frac{f(X)+\bar{r}(X)}{2}-\frac{\left\langle\hat{f}(X')\right\rangle_{\hat{h}_{1}}+\left\langle\hat{r}(X')\right\rangle_{\hat{h}_{2}}}{2}\right)\right)} \\
&\times\left(\left(h_{1}^{B}(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_{1}^{B}(X)\right\rangle\right)-\left(S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)\right)\frac{\partial_{f(X)}K_{X}\left|\Psi(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}+ZS(X)\right)\right. \\
&+S_{1}(X,X,\theta-1)\left(\frac{\partial f(X)}{\partial S^{T}(X,\theta-1)}\right)^{3} \\
&\times\left(-\frac{\partial_{f(X)}K_{X}\left|\Psi(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}\left(h_{1}^{B}(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_{1}^{B}(X)\right\rangle\right)\right)\right. \\
&-\left(S_{2}^{B}(X)+S_{1}^{B}(X)\right)\frac{\partial_{f(X)}K_{X}\left|\Psi(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}\left(1+\frac{\frac{1}{f(X)+C_{0}}}{\frac{\partial_{f(X)}K_{X}\left|\Psi(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}}\right)\Bigg) \\
&\left.\left.+Z^{2}S(X)-\frac{S(X)\,\partial_{f(X)}\left(\frac{\hat{K}_{X}\left|\hat{\Psi}(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}\right)}{\left(f(X)+C_{0}\right)\frac{\hat{K}_{X}\left|\hat{\Psi}(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}}\right)\right\}+2S_{1}(X,X,\theta-1)\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}\frac{\partial^{2}f(X)}{\partial\left(S^{T}(X,\theta-2)\right)^{2}} \\
&\times\left(h_{1}^{B}(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_{1}^{B}(X)\right\rangle\right)-\left(S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)\right)\frac{\partial_{f(X)}K_{X}\left|\Psi(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}+Z\right)
\end{aligned}
$$

$$
\frac{\delta}{\delta\bar{f}(X,\theta-1)}g\simeq g\frac{\partial_{\bar{f}(X)}\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}{\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}
$$

and we consider:

$$
\frac{\delta}{\delta\hat{f}(X,\theta-1)}g\simeq 0
$$

$$
\begin{aligned}
&\frac{\delta}{\delta S^{T}(X,,\theta-2)}g \\
\simeq\;& \left(h_{1}^{B}(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_{1}^{B}(X)\right\rangle\right)-\left(S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)\right)\left(1+\frac{1}{f(X)+C_{0}}\right)\frac{\partial_{f(X)}K_{X}\left|\Psi(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}\right) \\
&\times\frac{\partial_{\bar{f}(X)}\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}{\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}
\end{aligned}
$$

$$
\frac{\delta}{\delta\bar{f}(X,\theta-1)}i\simeq -g\frac{\partial_{\bar{f}(X)}\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}{\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}
$$

$$
\begin{aligned}
&\frac{\delta}{\delta\hat{f}(X,\theta-1)}i \\
\simeq\;& Z\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_{X}\left|\hat{\Psi}(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}\right)S(X)}{\frac{\hat{K}_{X}\left|\hat{\Psi}(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}}\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\delta}{\delta S^T(X,,\theta-2)} i \\
\simeq & -\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)-\left(S_1^B(X,\theta-1)+S_2^B(X,\theta-1)\right)\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\left(1+\frac{1}{f(X)+C_0}\right)\right) \\
& \times\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}\frac{\partial f(X)}{\partial S^T(X,\theta-2)} \\
& +S(X)\left(Z^2S(X)-\frac{S(X)\partial_{f(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\right)}{(f(X)+C_0)\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}}\right)\left[\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\right]^2+i\frac{\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\frac{\partial f(X)}{\partial S^T(X,\theta-2)}}
\end{aligned}
$$

A4.2 Formula for the convergence domain

We set:

$$
X=\left(\begin{array}{c}\delta\bar{f}(X,\theta)\\ \delta\hat{f}(X,\theta)\\ \delta S^T(X,,\theta-1)\end{array}\right), M=\left[\begin{array}{ccc}a&0&c\\ d&-1&f\\ g&0&i\end{array}\right], \mathbf{M}=\left[\begin{array}{c}\bar{M}\\ \hat{M}\\ M^T\end{array}\right]
$$

In average, neglecting the non-local interactions, we write the system as:

$$
X(\theta)=MX(\theta-1)+X^t\mathbf{M}X
$$

and consider the squared distance to the fixed point:

$$
(X(\theta))^t X(\theta)
$$

The time variation of this distance is given by the dyamics:

$$
\begin{aligned}
& (X(\theta-1))^t X(\theta-1)-(X(\theta))^t X(\theta) \\
= & (X(\theta-1))^t\left(1-M^tM\right)(X(\theta-1)) \\
& -(X(\theta-1))^t M^t\left(X^t\mathbf{M}X\right)-\left(X^t\mathbf{M}X\right)^t MX(\theta-1)-\left(X^t\mathbf{M}X\right)^t\left(X^t\mathbf{M}X\right)
\end{aligned}
$$

To the first approximation the range of the stability zone is given by the equation:

$$
\begin{aligned}
0 < & (X(\theta-1))^t\left(1-M^tM\right)(X(\theta-1)) \\
& -(X(\theta-1))^t M^t\left(X^t\mathbf{M}X\right)-\left(X^t\mathbf{M}X\right)^t MX(\theta-1)-\left(X^t\mathbf{M}X\right)^t\left(X^t\mathbf{M}X\right)
\end{aligned}
$$

To obtain some order of magnitude, we replace $X(\theta-1)$ by X and:

$$
M^tM\rightarrow\left\langle\lambda^2\right\rangle
$$

the average of squared eigenvalues, and:

$$
M\rightarrow\langle\lambda\rangle
$$

the average of eigenvalues. We obtain:

$$
x1(xax)+x2(xa2x)
$$

$$
x1(xax)+x2(xa2x)
$$

$$
0<\left(1-\left\langle\lambda^2\right\rangle\right)X^tX-\langle\lambda\rangle\left(X^t\left(X^t\mathbf{M}X\right)+\left(X^t\mathbf{M}X\right)^t X\right)-\left(X^t\mathbf{M}X\right)^t\left(X^t\mathbf{M}X\right)
$$

and this lead to the approximation:

$$\left(1-\left\langle\lambda^2\right\rangle\right)X^tX-\left(X^t\mathbf{M}X\right)^t\left(X^t\mathbf{M}X\right)>0$$

and the fluctuations are stable in the range:

$$\frac{\left(X^t\mathbf{M}X\right)^t\left(X^t\mathbf{M}X\right)}{X^tX}<1-\left\langle\lambda^2\right\rangle$$

given that:

$$\left(X^t\mathbf{M}X\right)=X^t\begin{bmatrix}X^t\bar{M}X\\X^t\hat{M}X\\X^tM^TX\end{bmatrix}$$

$$\frac{2\left\langle\lambda\right\rangle\delta\mathbf{S}^t\left[\delta S^T\left(X\right)M^T+\delta\hat{S}\left(X',X\right)\hat{M}+\delta\bar{S}\left(X',X\right)\bar{M}\right]\delta\mathbf{S}+\left(\delta\mathbf{S}^t\bar{M}\delta\mathbf{S}\right)^2+\left(\delta\mathbf{S}^t\hat{M}\delta\mathbf{S}\right)^2+\left(\delta\mathbf{S}^tM^T\delta\mathbf{S}\right)^2}{\left(\delta\bar{S}\left(X',X\right)\right)^2+\left(\delta\hat{S}\left(X',X\right)\right)^2+\left(\delta S^T\left(X\right)\right)^2}<1-\left\langle\lambda^2\right\rangle$$

where:

$$\delta\mathbf{S}=\begin{pmatrix}\delta\bar{S}\left(X',X\right)\\\delta\hat{S}\left(X',X\right)\\\delta S^T\left(X\right)\end{pmatrix}$$

For relatively large fluctuations, this becomes:

$$\frac{\left(\delta\mathbf{S}^t\bar{M}\delta\mathbf{S}\right)^2+\left(\delta\mathbf{S}^t\hat{M}\delta\mathbf{S}\right)^2+\left(\delta\mathbf{S}^tM^T\delta\mathbf{S}\right)^2}{\left(\delta\bar{S}\left(X',X\right)\right)^2+\left(\delta\hat{S}\left(X',X\right)\right)^2+\left(\delta S^T\left(X\right)\right)^2}<1-\left\langle\lambda^2\right\rangle \tag{196}$$

Given that we have neglected the role of investors we can set $\delta\hat{S}\left(X',X\right)=0$ and this writes:

$$\frac{\left(\delta\mathbf{S}^t\bar{M}\delta\mathbf{S}\right)^2+\left(\delta\mathbf{S}^t\hat{M}\delta\mathbf{S}\right)^2+\left(\delta\mathbf{S}^tM^T\delta\mathbf{S}\right)^2}{\left(\delta\bar{S}\left(X',X\right)\right)^2+\left(\delta S^T\left(X\right)\right)^2}<1-\left\langle\lambda^2\right\rangle \tag{197}$$

A4.3 Estimation of the convergence domain

In the system with banks, we can consider that $S^T\left(X,\theta-1\right)\rightarrow 1$. Under decreasing return to scale, this corresponds to consider that $\left|\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right|>>1$. We can thus estimate the corrections by keeping the higher order terms for each derivatives. We disregard terms involving $\delta\hat{f}$ since we neglected impact of investors fluctuations, that will be considered independently. This amount to consider that:

$$\begin{aligned}\bar{M}&=\begin{bmatrix}\frac{\delta}{\delta\bar{f}(X,\theta-1)}a&0&\frac{\bar{h}(X',X)}{2}\frac{\delta}{\delta\bar{f}(X,\theta-1)}c\\0&0&0\\\frac{\delta}{\delta S^T(X,,\theta-2)}a&0&\frac{\bar{h}(X',X)}{2}\frac{\delta}{\delta S^T(X,,\theta-2)}c3\end{bmatrix},\hat{M}=\begin{bmatrix}\frac{\hat{h}(X',X)}{\bar{h}(X',X)}\frac{\delta}{\delta\bar{f}(X,\theta-1)}d&0&\frac{\hat{h}(X',X)}{2}\frac{\delta}{\delta\bar{f}(X,\theta-1)}f\\0&0&0\\\frac{\hat{h}(X',X)}{\bar{h}(X',X)}\frac{\delta}{\delta S^T(X,,\theta-2)}d&0&\frac{\hat{h}(X',X)}{2}\frac{\delta}{\delta S^T(X,,\theta-2)}f\end{bmatrix}\\M^T&=0\end{aligned}$$

and the coefficients are approximated below.

We assume $S_1^B(X,\theta-1)+S_2^B(X,\theta-1)>>1$ due to bank loans:

$$\begin{aligned}
&\frac{\delta}{\delta S^T(X,,\theta-2)}c \\
\simeq\; & \left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3 \\
&\times\left\{h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)\right. \\
&\times\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)-\left(\left(S_1^B(X)+S_2^B(X)+S(X)\right)\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}-\frac{\hat{h}(X)}{2}S(X)\right)\right) \\
&+S_1^B(X,X)\left(\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\right. \\
&\times\left(\left(\left(S_1^B(X)+S_2^B(X)\right)\left(1+\frac{\frac{1}{f(X)+C_0}}{\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}}\right)+\frac{\frac{1}{f(X)+C_0}}{\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}}S(X)\right)\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}-h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\right.\right.\right. \\
&+\left.\left.\left(\frac{\hat{h}(X)}{2}-\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\right)^2S(X)\right)\right\} \\
&+c\frac{\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\frac{\partial f(X)}{\partial S^T(X,\theta-2)}}+S_1^B(X,X)\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\left\{h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)\right. \\
&\left.-\left(S_1^B(X,\theta-1)+S_2^B(X,\theta-1)+S(X)\right)\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}+\frac{\hat{h}(X)}{2}S(X)\right\}\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}
\end{aligned}$$

$$\begin{aligned}
&\frac{\delta}{\delta S^T(X,,\theta-2)}f \\
\simeq\; & S_1(X,X,\theta-1)\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3\frac{\hat{h}(X)}{2} \\
&\times\left(\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)-\left(S_1^B(X,\theta-1)+S_2^B(X,\theta-1)\right)\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}+ZS(X)\right)\right. \\
&+S_1(X,X,\theta-1)\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3 \\
&\times\left(-\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)-\left(S_2^B(X,\theta-1)+S_1^B(X,\theta-1)\right)\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\left(1+\right.\right.\right. \\
&\left.\left.\left.+Z^2S(X)-\frac{S(X)\,\partial_{f(X)}\left(\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}\right)}{(f(X)+C_0)\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}\right)\right)\right\}+2S_1(X,X,\theta-1)\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2} \\
&\times\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)-\left(S_1^B(X,\theta-1)+S_2^B(X,\theta-1)\right)\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}+Z\right)
\end{aligned}$$

and given that:

$$\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}<<1$$

$$\frac{\delta}{\delta \bar{f}(X,\theta-1)}c=\left(\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}\right)^{2} h_{1}^{B}(X)\left(\left\langle \bar{h}(X)\right\rangle +\left\langle \hat{h}_{1}^{B}(X)\right\rangle \right)$$

$$\begin{aligned}\frac{\delta}{\delta \bar{f}(X,\theta-1)}f &\simeq -S_{1}(X,X,\theta-1)\left(S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)\right)\\ &\times\left[\frac{\partial f(X)}{\partial S^{T}(X,\theta-1)}\right]^{2}\frac{\partial_{\bar{f}(X)}\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}{\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}\frac{\partial_{f(X)}K_{X}\left|\Psi(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}<<1\end{aligned}$$

$$\begin{aligned}&\frac{\delta}{\delta S^{T}(X,\theta-2)}a\\ \simeq&\left(S_{1}^{B}(X,X)\left(h_{1}^{B}(X)\left(\left\langle \bar{h}(X)\right\rangle +\left\langle \hat{h}_{1}^{B}(X)\right\rangle \right)-\left(S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)\right)\frac{\partial_{f(X)}K_{X}\left|\Psi(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}\right)\right.\\ &\left.+\left(S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)\right)h_{1}^{B}(X)\left(\left\langle \bar{h}(X)\right\rangle +\left\langle \hat{h}_{1}^{B}(X)\right\rangle \right)\right)\left(\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}\right)^{2}\frac{\partial_{f(X)}K_{X}\left|\Psi(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}\end{aligned}$$

$$\begin{aligned}&\frac{\delta}{\delta \bar{f}(X,\theta-1)}a\\ \simeq&\left(S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)\right)\\ &\times\left(\frac{\partial_{\bar{f}(X)}\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}{\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}S_{1}^{B}(X,X)+h_{1}^{B}(X)\left(\left\langle \bar{h}(X)\right\rangle +\left\langle \hat{h}_{1}^{B}(X)\right\rangle \right)\right)\frac{\partial_{\bar{f}(X)}\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}{\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}\frac{\partial f(X)}{\partial S^{T}(X,\theta-1)}\end{aligned}$$

$$\begin{aligned}&\frac{\delta}{\delta S^{T}(X,,\theta-2)}d\\ \simeq&\left(\left(\frac{\hat{h}(X)}{2}+\frac{\left(1-\frac{1}{f(X)+C_{0}}\right)h_{1}^{B}(X)\left(\left\langle \bar{h}(X)\right\rangle +\left\langle \hat{h}_{1}^{B}(X)\right\rangle \right)}{S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)}\right)\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}+\frac{\frac{\partial^{2}f(X)}{\partial\left(S^{T}(X,\theta-2)\right)^{2}}}{\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}}\right)d\\ =&\left(\left(\frac{\hat{h}(X)}{2}+\frac{\left(1-\frac{1}{f(X)+C_{0}}\right)h_{1}^{B}(X)\left(\left\langle \bar{h}(X)\right\rangle +\left\langle \hat{h}_{1}^{B}(X)\right\rangle \right)}{S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)}\right)\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}+\frac{\frac{\partial^{2}f(X)}{\partial\left(S^{T}(X,\theta-2)\right)^{2}}}{\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}}\right)\\ &\times S_{1}(X,X,\theta-1)\left(S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)\right)\frac{\partial f(X)}{\partial S^{T}(X,\theta-1)}\frac{\partial_{\bar{f}(X)}\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}{\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}\\ <&<1\end{aligned}$$

$$\begin{aligned}&\frac{\delta}{\delta \bar{f}(X,\theta-1)}d\\ \simeq&\ S_{1}(X,X,\theta-1)\left(S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)\right)\frac{\partial f(X)}{\partial S^{T}(X,\theta-1)}\left(\frac{\partial_{\bar{f}(X)}\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}{\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}\right)^{2}<<1\end{aligned}$$

A4.3.1 Case 1: $S^{T}(X)\frac{1}{f(X)+C_{0}}>h_{1}^{B}(X)\left(\left\langle \bar{h}(X)\right\rangle +\left\langle \hat{h}_{1}^{B}(X)\right\rangle \right)$

This case corresponds to a marginal return above a minimal value $\frac{1}{f_{0}(X)+C_{0}}$. This minimal value can be lower than 1 since due to loans, $S^{T}(X)\rightarrow 1$ and $h_{1}^{B}(X)\left(\left\langle \bar{h}(X)\right\rangle +\left\langle \hat{h}_{1}^{B}(X)\right\rangle \right)<\frac{1}{4}$ (with uncertainty, this is in general lower than $\frac{1}{8}$).

Derivation of $\frac{\delta}{\delta S^T(X,,\theta-2)}c$

When $S^T(X)\frac{1}{f(X)+C_0}>1$:

$$S^T(X)=\left(S_1^B(X,\theta-1)+S_2^B(X,\theta-1)+S(X)\right)$$

$$\begin{aligned}
&\frac{\delta}{\delta S^T(X,,\theta-2)}c\\
\simeq\;&\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\\
&\times\left\{-h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)S^T(X)\right.\\
&+S_1^B(X,X)\left(\left(\left(S_1^B(X,\theta-1)+S_2^B(X,\theta-1)\right)\left(1+\frac{\frac{1}{f(X)+C_0}}{\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}}\right)+\frac{\frac{1}{f(X)+C_0}}{\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}}S(X)\right)\right.\\
&\left.\left.\times\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}+\left(\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\right)S(X)\right)\right\}\\
&+c\frac{\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\frac{\partial f(X)}{\partial S^T(X,\theta-2)}}+S_1^B(X,X)\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\left\{h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)\right.\\
&\left.-S^T(X)\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}+\frac{\hat{h}(X)}{2}S(X)\right\}\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}
\end{aligned}$$

and this rewrites:

$$\begin{aligned}
&\frac{\delta}{\delta S^T(X,,\theta-2)}c\\
\simeq\;&\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\\
&\times\left\{-h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)S^T(X)\right.\\
&\left.+\,S_1^B(X,X)\left(S^T(X)\left(1+\frac{\frac{1}{f(X)+C_0}}{\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}}\right)\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\right)\right\}\\
&+c\frac{\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\frac{\partial f(X)}{\partial S^T(X,\theta-2)}}-S_1^B(X,X)\frac{\partial f(X)}{\partial S^T(X,\theta-1)}S^T(X)\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}
\end{aligned}$$

Given that:

$$\begin{aligned}
&c\frac{\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\frac{\partial f(X)}{\partial S^T(X,\theta-2)}}\\
\simeq\;&-S_1^B(X,X)\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\left(S_1^B(X,\theta-1)+S_2^B(X,\theta-1)+S(X)\right)\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\\
&\times\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}\\
=\;&-S_1^B(X,X)S^T(X)\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}
\end{aligned}$$

We obtain:

$$
\begin{aligned}
&\frac{\delta}{\delta S^{T}\left(X,,\theta-2\right)}c\\
\simeq\quad & \left(\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-1\right)}\right)^{3}\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}\\
&\times\left\{-h_{1}^{B}\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_{1}^{B}\left(X\right)\right\rangle\right)S^{T}\left(X\right)\right.\\
&\left.+S_{1}^{B}\left(X,X\right)\left(S^{T}\left(X\right)\left(1+\frac{\frac{1}{f(X)+C_{0}}}{\frac{\partial_{f(X)}K_{X}|\Psi(X)|^{2}}{K_{X}|\Psi(X)|^{2}}}\right)\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}\right.\right\}\\
&-2S_{1}^{B}\left(X,X\right)\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-1\right)}S^{T}\left(X\right)\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}\frac{\partial^{2}f\left(X\right)}{\partial\left(S^{T}\left(X,\theta-2\right)\right)^{2}}
\end{aligned}
$$

and in an expanded form:

$$
\begin{aligned}
&\frac{\delta}{\delta S^{T}\left(X,,\theta-2\right)}c\\
\simeq\quad & \left(\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-1\right)}\right)^{3}\left(\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}\right)^{2}S_{1}^{B}\left(X,X\right)S^{T}\left(X\right)\\
&\times\left\{\left(1+\frac{\frac{1}{f(X)+C_{0}}}{\frac{\partial_{f(X)}K_{X}|\Psi(X)|^{2}}{K_{X}|\Psi(X)|^{2}}}\right)-\frac{\frac{h_{1}^{B}(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_{1}^{B}(X)\right\rangle\right)}{S_{1}^{B}(X,X)}+2\frac{\frac{\partial^{2}f(X)}{\partial\left(S^{T}(X,\theta-2)\right)^{2}}}{\left(\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}\right)^{2}}}{\frac{\partial_{f(X)}K_{X}|\Psi(X)|^{2}}{K_{X}|\Psi(X)|^{2}}}\right\}
\end{aligned}
$$

Derivation of $\frac{\delta}{\delta S^{T}(X,,\theta-2)}f$

$$
\begin{aligned}
&\frac{\delta}{\delta S^{T}\left(X,,\theta-2\right)}f\\
\simeq\quad & S_{1}\left(X,X,\theta-1\right)\left(\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-1\right)}\right)^{3}\frac{\hat{h}\left(X\right)}{2}\left(-S^{T}\left(X\right)\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}\right)\\
&+S_{1}\left(X,X,\theta-1\right)\left(\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-1\right)}\right)^{3}\\
&\times\left(+\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}\left(\left(S^{T}\left(X\right)\right)\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}\left(1+\frac{\frac{1}{f(X)+C_{0}}}{\frac{\partial_{f(X)}K_{X}|\Psi(X)|^{2}}{K_{X}|\Psi(X)|^{2}}}\right)\right)\right)\Bigg\}\\
&+2S_{1}\left(X,X,\theta-1\right)\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-2\right)}\frac{\partial^{2}f\left(X\right)}{\partial\left(S^{T}\left(X,\theta-2\right)\right)^{2}}\\
&\times\left(h_{1}^{B}\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_{1}^{B}\left(X\right)\right\rangle\right)-\left(S_{1}^{B}\left(X,\theta-1\right)+S_{2}^{B}\left(X,\theta-1\right)\right)\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}+Z\right)
\end{aligned}
$$

that writes:

$$\begin{aligned}
& \frac{\delta}{\delta S^{T}\left(X,,\theta-2\right)}f \\
\simeq\ & S_{1}\left(X,X,\theta-1\right)\left(\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-1\right)}\right)^{3}\frac{\hat{h}\left(X\right)}{2}\left(-S^{T}\left(X\right)\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}\right) \\
& +S_{1}\left(X,X,\theta-1\right)\left(\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-1\right)}\right)^{3} \\
& \times\left(\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}\left(S^{T}\left(X\right)\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}\left(1+\frac{\frac{1}{f(X)+C_{0}}}{\frac{\partial_{f(X)}K_{X}|\Psi(X)|^{2}}{K_{X}|\Psi(X)|^{2}}}\right)\right)\right)\Bigg\} \\
& +2S_{1}\left(X,X,\theta-1\right)\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-2\right)}\frac{\partial^{2}f\left(X\right)}{\partial\left(S^{T}\left(X,\theta-2\right)\right)^{2}}\left(-S^{T}\left(X\right)\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}\right)
\end{aligned}$$

Gathering the various terms yields:

$$\begin{aligned}
& \frac{\delta}{\delta S^{T}\left(X,,\theta-2\right)}f \\
\simeq\ & S_{1}\left(X,X\right)S^{T}\left(X\right)\left(\frac{\partial f\left(X\right)}{\partial S^{T}\left(X\right)}\right)^{3}\left(\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}\right)^{2} \\
& \times\left\{\left(1+\frac{\frac{1}{f(X)+C_{0}}}{\frac{\partial_{f(X)}K_{X}|\Psi(X)|^{2}}{K_{X}|\Psi(X)|^{2}}}\right)-\frac{\frac{\hat{h}(X)}{2}+2\frac{\frac{\partial^{2}f(X)}{\partial\left(S^{T}(X,\theta-2)\right)^{2}}}{\left(\frac{\partial f(X)}{\partial S^{T}(X,\theta-1)}\right)^{2}}}{\frac{\partial_{f(X)}K_{X}|\Psi(X)|^{2}}{K_{X}|\Psi(X)|^{2}}}\right\}
\end{aligned}$$

Estimation of $\frac{\delta}{\delta f(X,\theta-1)}c$ and $\frac{\delta}{\delta f(X,\theta-1)}d$

$$\frac{\delta}{\delta\bar{f}\left(X,\theta-1\right)}c=\left(\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-2\right)}\right)^{2}h_{1}^{B}\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_{1}^{B}\left(X\right)\right\rangle\right)$$

$$\begin{aligned}
& \frac{\delta}{\delta\bar{f}\left(X,\theta-1\right)}d \\
\simeq\ & S_{1}\left(X,X,\theta-1\right)\left(S_{1}^{B}\left(X,\theta-1\right)+S_{2}^{B}\left(X,\theta-1\right)\right)\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-1\right)}\left(\frac{\partial_{\bar{f}(X)}\bar{K}_{X}\left|\bar{\Psi}\left(X\right)\right|^{2}}{\bar{K}_{X}\left|\bar{\Psi}\left(X\right)\right|^{2}}\right)^{2}<<1
\end{aligned}$$

Estimation of $\frac{\delta}{\delta S^{T}(X,\theta-2)}a$ and $\frac{\delta}{\delta\bar{f}(X,\theta-1)}a$

$$\begin{aligned}
& \frac{\delta}{\delta S^{T}\left(X,\theta-2\right)}a \\
\simeq\ & -S_{1}^{B}\left(X,X\right)\left(S_{1}^{B}\left(X,\theta-1\right)+S_{2}^{B}\left(X,\theta-1\right)\right)\times\left(\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}\right)^{2}\left(\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-2\right)}\right)^{2}
\end{aligned}$$

$$\begin{aligned}
& \frac{\delta}{\delta\bar{f}\left(X,\theta-1\right)}a \\
\simeq\ & S_{1}^{B}\left(X,X\right)\left(S_{1}^{B}\left(X,\theta-1\right)+S_{2}^{B}\left(X,\theta-1\right)\right)\left(\frac{\partial_{\bar{f}(X)}\bar{K}_{X}\left|\bar{\Psi}\left(X\right)\right|^{2}}{\bar{K}_{X}\left|\bar{\Psi}\left(X\right)\right|^{2}}\right)^{2}\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-1\right)}
\end{aligned}$$

Estimation of the stability domain
Condition (196) becomes:

$$\begin{aligned}
&\left(1-\left\langle\lambda^2\right\rangle\right)\left(\left(\delta\bar{S}\left(X',X\right)\right)^2+\left(\delta S^T\left(X\right)\right)^2\right)\\
>&\left\{\frac{\frac{\left\langle\bar{h}(X,X)\right\rangle}{2}\delta c}{\delta S^T\left(X,,\theta-2\right)}\left(\delta S^T\left(X\right)\right)^2+\delta S^T\left(X\right)\left(\frac{\delta a}{\delta S^T\left(X,,\theta-2\right)}+\frac{\frac{\bar{h}(X',X)}{2}\delta c}{\delta\bar{f}\left(X,\theta-1\right)}\right)\delta\bar{S}\left(X\right)+\frac{\delta a}{\delta\bar{f}\left(X,\theta-1\right)}\left(\delta S^T\left(X\right)\right)^2\right\}^2\\
&+\left\{\frac{\left\langle\hat{h}\left(X',X\right)\right\rangle}{2}\left(\frac{\delta f}{\delta S^T\left(X,,\theta-2\right)}\right)\left(\delta S^T\left(X\right)\right)^2+\frac{\hat{h}\left(X',X\right)}{\bar{h}\left(X',X\right)}\frac{\delta d}{\delta\bar{f}\left(X,\theta-1\right)}\left(\delta\bar{S}\left(X\right)\right)^2\right\}^2
\end{aligned}$$

and this writes:

$$\begin{aligned}
1-\left\langle\lambda^2\right\rangle &> \left(\left\{S_1^B\left(X,X\right)\left(A\left(\delta S^T\left(X\right)\right)^2+B\left(\delta\bar{S}\left(X',X\right)\right)^2+\delta S^T\left(X\right)C\delta\bar{S}\left(X',X\right)\right)\right\}^2\right.\\
&\left.+\left\{S_1\left(X,X\right)\left(D\left(\delta S^T\left(X\right)\right)^2+L\left(\delta\bar{S}\left(X',X\right)\right)^2\right)\right\}^2\right)\\
&\times\frac{\left\{\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3\left(\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\right)^2 S^T\left(X\right)\right\}^2}{\left(\delta\bar{S}\left(X',X\right)\right)^2+\left(\delta S^T\left(X\right)\right)^2}
\end{aligned}$$

with:

$$\begin{aligned}
A &= \frac{\bar{h}\left(X\right)}{2}\left[\left(1+\frac{\frac{1}{f(X)+C_0}}{\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}}\right)-\frac{\frac{h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)}{S_1^B(X,X)}+2\frac{\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\left(\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\right)^2}}{\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}}\right]\\
B &= \frac{\left(S_1^B\left(X,\theta-1\right)+S_2^B\left(X,\theta-1\right)\right)}{\frac{\partial f(X)}{\partial S^T(X,\theta-2)}S^T\left(X\right)}\\
C &= \frac{\frac{\bar{h}(X',X)}{2}h_1^B\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_1^B\left(X\right)\right\rangle\right)}{\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\left(\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\right)^2 S^T\left(X\right)S_1^B\left(X,X\right)}-\frac{\left(S_1^B\left(X,\theta-1\right)+S_2^B\left(X,\theta-1\right)\right)}{\frac{\partial f(X)}{\partial S^T(X,\theta-2)}S^T\left(X\right)}\\
D &= \frac{\hat{h}\left(X\right)}{2}\left[\left(1+\frac{\frac{1}{f(X)+C_0}}{\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}}\right)-\frac{\frac{\hat{h}(X)}{2}+2\frac{\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^2}}{\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}}\right]\\
L &= \frac{\hat{h}\left(X',X\right)}{\bar{h}\left(X',X\right)}\frac{\left(S_1^B\left(X,\theta-1\right)+S_2^B\left(X,\theta-1\right)\right)\left(\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}\right)^2}{S^T\left(X\right)\left(\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\right)^2\left(\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\right)^2}
\end{aligned}$$

$$\lambda=\frac{1}{2}a+\frac{1}{2}i\pm\frac{1}{2}\sqrt{\left(a-i\right)^2+4cg}$$

$$a\simeq\left(S_1^B(X,\theta-1)+S_2^B(X,\theta-1)\right)\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}S_1^B(X,X)\frac{\partial f(X)}{\partial S^T(X,\theta-1)}$$

$$\begin{aligned}c&=S_1^B(X,X)\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\left\{h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)-S^T(X)\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\right.\\&\left.\left.+\left(\frac{\hat{h}(X)}{2}S(X)\right)\right\}\right\}\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\end{aligned}$$

$$i=\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)-S^T(X)\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}+\left(\frac{\hat{h}(X)}{2}S(X)\right)\right)\frac{\partial f(X)}{\partial S^T(X,\theta-2)}$$

$$\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}<<1$$

which implies:

$$g=\left(S_1^B(X,\theta-1)+S_2^B(X,\theta-1)\right)\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}<<1$$

$$a<<i$$

and:

$$\lambda\simeq\frac{1}{2}i\pm\frac{1}{2}i$$

$$\left\langle\lambda^2\right\rangle\simeq\frac{i^2}{2}$$

and:

$$1-\left\langle\lambda^2\right\rangle\simeq1-\left(S^T(X)\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\right)^2$$

A4.3.2 Case 2: $S^{TT}(X)\frac{1}{f(X)+C_0}<h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)$

Derivation of $\frac{\delta}{\delta S^T(X,,\theta-2)}c$

When $\frac{1}{f(X)+C_0}<1$:

$$\begin{aligned}&\frac{\delta}{\delta S^T(X,,\theta-2)}c\\\simeq&\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)\\&\times\left\{h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)+\frac{\hat{h}(X)}{2}S(X)\right.\\&\left.-S_1^B(X,X)\left(\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)-\left(\frac{\hat{h}(X)}{2}\right)^2S(X)\right)\right\}\\&+c\frac{\frac{\partial^2f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\frac{\partial f(X)}{\partial S^T(X,\theta-2)}}+S_1^B(X,X)\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\left\{h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)\right.\\&\left.+\frac{\hat{h}(X)}{2}S(X)\right\}\frac{\partial^2f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}\end{aligned}$$

We use that:

$$
\begin{aligned}
& c\frac{\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\frac{\partial f(X)}{\partial S^T(X,\theta-2)}} \\
\simeq\;& S_1^B(X,X)\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)+\frac{\hat{h}(X)}{2}S(X)\right)\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}
\end{aligned}
$$

and we obtain:

$$
\begin{aligned}
& \frac{\delta}{\delta S^T(X,,\theta-2)}c \\
\simeq\;& \left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3 h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right) \\
& \times\left\{h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)+\frac{\hat{h}(X)}{2}S(X)\right. \\
& \left.-S_1^B(X,X)\left(\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)-\left(\frac{\hat{h}(X)}{2}\right)^2S(X)\right)\right\} \\
& +2S_1^B(X,X)\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)+\frac{\hat{h}(X)}{2}S(X)\right)\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}
\end{aligned}
$$

Ultimately this yields:

$$
\begin{aligned}
& \frac{\delta}{\delta S^T(X,,\theta-2)}c \\
\simeq\;& \left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)+\frac{\hat{h}(X)}{2}S(X)\right) \\
& \times\left\{h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)+S_1^B(X,X)\frac{2\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^2}-\frac{\left(\frac{\hat{h}(X)}{2}\right)^2S(X)}{h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)+\frac{\hat{h}(X)}{2}S(X)}\right\} \\
\simeq\;& \left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)\right)h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)
\end{aligned}
$$

where we used that $S(X)<<1$ given the bank loans.

Derivation of $\frac{\delta}{\delta S^T(X,,\theta-2)}f$

$$
\begin{aligned}
& \frac{\delta}{\delta S^T(X,,\theta-2)}f \\
\simeq\;& S_1(X,X)\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3\frac{\hat{h}(X)}{2}\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)+\frac{\hat{h}(X)}{2}S(X)\right) \\
& +S_1(X,X)\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3\left(-\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)+\left(\frac{\hat{h}(X)}{2}\right)^2S(X)\right) \\
& +2S_1(X,X)\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)+\frac{\hat{h}(X)}{2}S(X)\right)
\end{aligned}
$$

that is factored as:

$$\begin{aligned}
&\frac{\delta}{\delta S^{T}\left(X,,\theta-2\right)}f\\
\simeq\ & S_{1}\left(X,X\right)\left(\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-1\right)}\right)^{3}\left(h_{1}^{B}\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_{1}^{B}\left(X\right)\right\rangle\right)+\frac{\hat{h}\left(X\right)}{2}S\left(X\right)\right)\\
&\times\left(\frac{\hat{h}\left(X\right)}{2}+\frac{\left(\frac{\hat{h}(X)}{2}\right)^{2}S\left(X\right)-\frac{\partial_{f(X)}K_{X}\left|\Psi(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}h_{1}^{B}\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_{1}^{B}\left(X\right)\right\rangle\right)}{h_{1}^{B}\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_{1}^{B}\left(X\right)\right\rangle\right)+\frac{\hat{h}(X)}{2}S\left(X\right)}+2\frac{\frac{\partial^{2}f(X)}{\partial\left(S^{T}(X,\theta-2)\right)^{2}}}{\left(\frac{\partial f(X)}{\partial S^{T}(X,\theta-1)}\right)^{2}}\right)\\
\simeq\ & S_{1}\left(X,X\right)\left(\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-1\right)}\right)^{3}\left(h_{1}^{B}\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_{1}^{B}\left(X\right)\right\rangle\right)\right)\\
&\times\left(\frac{\hat{h}\left(X\right)}{2}-\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}+2\frac{\frac{\partial^{2}f(X)}{\partial\left(S^{T}(X,\theta-2)\right)^{2}}}{\left(\frac{\partial f(X)}{\partial S^{T}(X,\theta-1)}\right)^{2}}\right)
\end{aligned}$$

Estimation of $\frac{\delta}{\delta\bar{f}(X,\theta-1)}c$ and $\frac{\delta}{\delta\bar{f}(X,\theta-1)}d$

$$\frac{\delta}{\delta\bar{f}\left(X,\theta-1\right)}c=\left(\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-2\right)}\right)^{2}h_{1}^{B}\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_{1}^{B}\left(X\right)\right\rangle\right)$$

$$\begin{aligned}
&\frac{\delta}{\delta\bar{f}\left(X,\theta-1\right)}d\\
\simeq\ & S_{1}\left(X,X,\theta-1\right)\left(S_{1}^{B}\left(X,\theta-1\right)+S_{2}^{B}\left(X,\theta-1\right)\right)\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-1\right)}\left(\frac{\partial_{\bar{f}(X)}\bar{K}_{X}\left|\bar{\Psi}\left(X\right)\right|^{2}}{\bar{K}_{X}\left|\bar{\Psi}\left(X\right)\right|^{2}}\right)^{2}<<1
\end{aligned}$$

Estimation of $\frac{\delta}{\delta S^{T}(X,\theta-2)}a$ and $\frac{\delta}{\delta\bar{f}(X,\theta-1)}a$

$$\begin{aligned}
&\frac{\delta}{\delta S^{T}\left(X,\theta-2\right)}a\\
\simeq\ & h_{1}^{B}\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_{1}^{B}\left(X\right)\right\rangle\right)\\
&\times\left(S_{1}^{B}\left(X,X\right)+\left(S_{1}^{B}\left(X,\theta-1\right)+S_{2}^{B}\left(X,\theta-1\right)\right)\left(\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-2\right)}\right)^{2}\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}\right)\\
\rightarrow\ & h_{1}^{B}\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_{1}^{B}\left(X\right)\right\rangle\right)\left(\left(S_{1}^{B}\left(X,\theta-1\right)+S_{2}^{B}\left(X,\theta-1\right)\right)\left(\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-2\right)}\right)^{2}\frac{\partial_{f(X)}K_{X}\left|\Psi\left(X\right)\right|^{2}}{K_{X}\left|\Psi\left(X\right)\right|^{2}}\right)
\end{aligned}$$

$$\begin{aligned}
&\frac{\delta}{\delta\bar{f}\left(X,\theta-1\right)}a\\
\simeq\ & \left(S_{1}^{B}\left(X,\theta-1\right)+S_{2}^{B}\left(X,\theta-1\right)\right)\\
&\times\left(h_{1}^{B}\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_{1}^{B}\left(X\right)\right\rangle\right)\right)\frac{\partial_{\bar{f}(X)}\bar{K}_{X}\left|\bar{\Psi}\left(X\right)\right|^{2}}{\bar{K}_{X}\left|\bar{\Psi}\left(X\right)\right|^{2}}\frac{\partial f\left(X\right)}{\partial S^{T}\left(X,\theta-1\right)}
\end{aligned}$$

Estimation of the stability domain

Condition (196) becomes:

$$
\begin{aligned}
&\left(1-\left\langle\lambda^2\right\rangle\right)\left(\left(\delta\bar{S}\left(X',X\right)\right)^2+\left(\delta S^T\left(X\right)\right)^2\right)\\
>\quad&\left\{\frac{\left(\delta S^T\left(X\right)\right)^2\frac{\left\langle\bar{h}(X,X)\right\rangle}{2}\delta c}{\delta S^T\left(X,,\theta-2\right)}+\delta S^T\left(X\right)\left(\frac{\delta a}{\delta S^T\left(X,,\theta-2\right)}+\frac{\frac{\bar{h}(X',X)}{2}\delta c}{\delta\bar{f}\left(X,\theta-1\right)}\right)\delta\bar{S}\left(X\right)+\frac{\left(\delta S^T\left(X\right)\right)^2\delta a}{\delta\bar{f}\left(X,\theta-1\right)}\right\}^2\\
&+\left\{\frac{\left\langle\hat{h}\left(X',X\right)\right\rangle}{2}\frac{\delta f}{\delta S^T\left(X,,\theta-2\right)}\left(\delta S^T\left(X\right)\right)^2+\frac{\hat{h}\left(X',X\right)}{\bar{h}\left(X',X\right)}\frac{\delta d}{\delta\bar{f}\left(X,\theta-1\right)}\left(\delta\bar{S}\left(X\right)\right)^2\right\}^2
\end{aligned}
$$

which can be developed as:

$$
\begin{aligned}
1-\left\langle\lambda^2\right\rangle\quad>\quad&\left(\left\{S_1^B\left(X,X\right)\left(\frac{\bar{h}\left(X\right)}{2}A\left(\delta S^T\left(X\right)\right)^2+B\left(\delta\bar{S}\left(X',X\right)\right)^2+\delta S^T\left(X\right)C\delta\bar{S}\left(X',X\right)\right)\right\}^2\right.\\
&+\left\{S_1\left(X,X\right)\left(\left(\frac{\hat{h}\left(X\right)}{2}D\left(\delta S^T\left(X\right)\right)^2\right)+L\left(\delta\bar{S}\left(X',X\right)\right)^2\right)\right\}^2\\
&\times\frac{\left\{\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3\left(h_1^B\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_1^B\left(X\right)\right\rangle\right)+\frac{\hat{h}(X)}{2}S\left(X\right)\right)\right\}^2}{\left(\delta\bar{S}\left(X',X\right)\right)^2+\left(\delta S^T\left(X\right)\right)^2}
\end{aligned}
$$

where the coefficients are defined by:

$$A = h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle + \left\langle \hat{h}_1^B(X)\right\rangle\right) + S_1^B(X,X)\frac{2\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^2} - \frac{\left(\frac{\hat{h}(X)}{2}\right)^2 S(X)}{h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle + \left\langle \hat{h}_1^B(X)\right\rangle\right) + \frac{\hat{h}(X)}{2}S(X)}$$

$$\simeq h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle + \left\langle \hat{h}_1^B(X)\right\rangle\right) + S_1^B(X,X)\frac{2\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^2}$$

$$B = \frac{\left(S_1^B(X,\theta-1) + S_2^B(X,\theta-1)\right)\left(h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle + \left\langle \hat{h}_1^B(X)\right\rangle\right)\right)\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}}{\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^2\left(h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle + \left\langle \hat{h}_1^B(X)\right\rangle\right) + \frac{\hat{h}(X)}{2}S(X)\right)}$$

$$\simeq \frac{\left(S_1^B(X,\theta-1) + S_2^B(X,\theta-1)\right)\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}}{\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^2}$$

$$C = \frac{h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle + \left\langle \hat{h}_1^B(X)\right\rangle\right)\left(\frac{\bar{h}\left(X',X\right)}{2} + \left(S_1^B(X,\theta-1) + S_2^B(X,\theta-1)\right)\frac{\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}}{\frac{\partial f(X)}{\partial S^T(X,\theta-1)}}\right)}{\frac{\partial f(X)}{\partial S^T(X,\theta-2)}h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle + \left\langle \hat{h}_1^B(X)\right\rangle\right) + \frac{\hat{h}(X)}{2}S(X)}$$

$$\simeq \frac{\bar{h}(X',X)}{2\frac{\partial f(X)}{\partial S^T(X,\theta-2)}} + \left(S_1^B(X,\theta-1) + S_2^B(X,\theta-1)\right)\frac{\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}}{\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^2}$$

$$D = S_1(X,X)\left(\frac{\hat{h}(X)}{2} + \frac{\left(\frac{\hat{h}(X)}{2}\right)^2 S(X) - \frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle + \left\langle \hat{h}_1^B(X)\right\rangle\right)}{h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle + \left\langle \hat{h}_1^B(X)\right\rangle\right) + \frac{\hat{h}(X)}{2}S(X)} + 2\frac{\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^2}\right)$$

$$\simeq S_1(X,X)\frac{\hat{h}(X)}{2} - \frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2} + 2\frac{\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^2}$$

$$L = \frac{\hat{h}(X',X)}{\bar{h}(X',X)}\frac{\left(S_1^B(X,\theta-1) + S_2^B(X,\theta-1)\right)\left(\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}\right)^2}{\left(\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\right)^2\left(h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle + \left\langle \hat{h}_1^B(X)\right\rangle\right) + \frac{\hat{h}(X)}{2}S(X)\right)}$$

$$\simeq \frac{\hat{h}(X',X)}{\bar{h}(X',X)}\frac{\left(S_1^B(X,\theta-1) + S_2^B(X,\theta-1)\right)\left(\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}\right)^2}{\left(\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\right)^2\left(h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle + \left\langle \hat{h}_1^B(X)\right\rangle\right)\right)}\frac{\hat{h}(X',X)}{\bar{h}(X',X)}$$

Replacing the various quantities by their equilibrium static values yields:

$$A \simeq h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle + \left\langle \hat{h}_1^B(X)\right\rangle\right) + S_1^B(X,X)\frac{2\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^2}$$

$$B \simeq \frac{\left(S_1^B(X,\theta-1) + S_2^B(X,\theta-1)\right)\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}}{\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^2}$$

and:

$$C \simeq \frac{\bar{h}(X',X)}{2\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}} + \left(S_1^B(X) + S_2^B(X)\right)\frac{\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}}{\left(\frac{\partial f(X)}{\partial S^{T}(X,\theta-1)}\right)^2}$$

$$D \simeq S_1(X,X)\frac{\hat{h}(X)}{2} - \frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2} + 2\frac{\frac{\partial^2 f(X)}{\partial\left(S^{T}(X,\theta-2)\right)^2}}{\left(\frac{\partial f(X)}{\partial S^{T}(X,\theta-1)}\right)^2}$$

$$L \simeq \frac{\hat{h}(X',X)}{\bar{h}(X',X)}\frac{\left(S_1^B(X)+S_2^B(X)\right)\left(\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}\right)^2}{\left(\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}\right)^2\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)\right)}$$

A 4.4 Quadratic order dynamics for investors

$$\begin{pmatrix}\delta\hat{S}(X',X,\theta)\\ \delta S(X,,\theta-1)\end{pmatrix} = \begin{bmatrix}\alpha & \frac{\hat{h}(X',X)}{2}c\\ \frac{h}{\frac{\hat{h}(X',X)}{2}} & \beta\end{bmatrix}\begin{pmatrix}\delta\hat{S}(X',X,\theta-1)\\ \delta S(X,,\theta-2)\end{pmatrix} + \begin{pmatrix}\delta\hat{S}(X',X,\theta)\\ \delta S(X,,\theta-1)\end{pmatrix}^t\begin{bmatrix}\hat{M}\\ M\end{bmatrix}\begin{pmatrix}\delta\hat{S}(X',X,\theta)\\ \delta S(X,\theta-1)\end{pmatrix}$$

$$\hat{M} = \begin{bmatrix}\frac{\delta}{\delta\hat{f}(X,\theta-1)}\alpha & \frac{\hat{h}(X',X)}{2}\frac{\delta}{\delta\hat{f}(X,\theta-1)}c\\ \frac{\delta}{\delta S(X,,\theta-1)}\alpha & \frac{\hat{h}(X',X)}{2}\frac{\delta}{\delta S(X,,\theta-1)}c\end{bmatrix}, M = \begin{bmatrix}\frac{1}{\frac{\hat{h}(X',X)}{2}}\frac{\delta}{\delta\hat{f}(X,\theta-1)}h & \frac{\delta}{\delta\hat{f}(X,\theta-1)}\beta\\ \frac{1}{\frac{\hat{h}(X',X)}{2}}\frac{\delta}{\delta S(X,,\theta-1)}h & \frac{\delta}{\delta S(X,,\theta-1)}\beta\end{bmatrix}$$

The condition for stability becomes:

$$\begin{aligned}&1-\left\langle\lambda^2\right\rangle\\ > &\left\{\frac{\delta\alpha}{\delta\hat{f}(X)}\left(\delta\hat{S}(X)\right)^2 + \delta\hat{S}(X)\left(\frac{\frac{\hat{h}(X',X)}{2}\delta c}{\delta\hat{f}(X)} + \frac{\delta\alpha}{\delta S(X)}\right)\delta S(X) + \frac{\frac{\hat{h}(X',X)}{2}\delta c}{\delta S(X,\theta-1)}\left(\delta S(X)\right)^2\right\}^2\\ &+\left\{\frac{\delta h}{\frac{\hat{h}(X',X)}{2}\delta\hat{f}(X)}\left(\delta\hat{S}(X)\right)^2\right.\\ &\left.+\delta\hat{S}(X)\left(\frac{\delta h}{\frac{\hat{h}(X',X)}{2}\delta S(X,,\theta-1)} + \frac{\delta\beta}{\delta\hat{f}(X)}\right)\delta S(X) + \frac{\delta\beta}{\delta S(X,,\theta-1)}\left(\delta S(X)\right)^2\right\}^2\end{aligned}$$

where the coefficients are given in Gosselin and Lotz (2025a):

$$\begin{aligned}\alpha = &\left(\frac{\hat{S}(X)}{\left(1-\left(\hat{S}(X)\right)\right)}\left(\frac{1}{2\left(1+\frac{\Delta\hat{f}(X')+\Delta\hat{r}(X')}{2}\right)} - \frac{\frac{\partial\hat{K}_{X'}\left|\hat{\Psi}(X')\right|^2}{\partial\hat{f}(X,\theta-1)}}{\hat{K}_{X'}\left|\hat{\Psi}(X')\right|^2}\right)\right.\\ &\left.-\frac{\hat{S}_1(X)}{\left(1-\left(\hat{S}_1(X')\right)\right)}\left(\frac{1}{1+\Delta\hat{f}(X)} - \frac{\frac{\partial\hat{K}_{X'}\left|\hat{\Psi}(X')\right|^2}{\partial\hat{f}(X,\theta-1)}}{\hat{K}_{X'}\left|\hat{\Psi}(X')\right|^2}\right)\right)\left(\hat{f}(X)-\bar{r}\right)\\ &+\frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)}S_1(X,X,\theta-1)\frac{\partial f(X)}{\partial S(X,\theta-1)}\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\right)S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}}\end{aligned} \tag{198}$$

$$\beta = \left(\frac{\frac{\hat{h}(X)}{2}S(X)}{1+\left(\hat{h}(X)\left(\frac{f(X)+\bar{r}(X)}{2}-\frac{\left\langle\hat{f}(X')\right\rangle_{\hat{h}_1}+\langle\hat{r}(X')\rangle_{\hat{h}_2}}{2}\right)\right)}+\frac{\partial_{f(X)}\left(\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}\right)\frac{\partial f(X)}{\partial S(X,\theta-2)}$$

$$\simeq \left(\frac{\hat{h}(X)}{2}S(X)+\frac{\partial_{f(X)}\left(\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}\right)\frac{\partial f(X)}{\partial S(X,\theta-2)}$$

$$c = \left\{-\frac{\left(1-\left\langle\hat{S}(X')\right\rangle\right)\left(\left\langle\hat{f}(X')\right\rangle-\bar{r}\right)}{1-\left\langle\hat{S}_1(X')\right\rangle}\frac{\left\langle\hat{h}(X',X)\right\rangle\langle h(X)\rangle}{4}\right.$$
$$+\frac{\frac{h(X)}{4}\left(1+\hat{h}(X)\right)S_1(X)(f(X)-\bar{r})}{1+\left(\hat{h}(X)\left(f(X)-\frac{\left\langle\hat{f}(X')\right\rangle_{\hat{h}_1}+\langle\hat{r}(X')\rangle_{\hat{h}_2}}{2}\right)+\frac{h(X)}{2}(f(X)-\bar{r}(X))\right)}$$
$$\left.+S_1(X)\left(\frac{\frac{\hat{h}(X)}{2}S(X)}{1+\left(\hat{h}(X)\left(\frac{f(X)+\bar{r}(X)}{2}-\frac{\left\langle\hat{f}(X')\right\rangle_{\hat{h}_1}+\langle\hat{r}(X')\rangle_{\hat{h}_2}}{2}\right)\right)}+\frac{\partial_{f(X)}\left(\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}\right)\frac{\partial f(X)}{\partial S(X,\theta-2)}\right\}$$
$$\times\frac{\partial f(X)}{\partial S(X,\theta-1)}$$

$$h = \frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}$$

A4.4.1 Estimation of coefficients derivatives

The variations of these coefficients are obtained by using the following formula derived in (2025a).

$$\begin{aligned}&\delta\hat{S}_1(X',X,\theta-1)\\ \rightarrow\quad&\frac{\hat{h}(X',X)}{2}\left(1+\left(\delta\hat{f}(X')-\frac{\langle h(X)\rangle}{2}\delta f(X)\right)\right)\end{aligned}$$

$$\delta\hat{S}_1(X,\theta-1)\simeq\left(1-\frac{\frac{\partial\hat{K}_{X'}|\hat{\Psi}(X')|^2}{\partial\hat{f}(X,\theta-1)}}{\hat{K}_{X'}\left|\hat{\Psi}(X')\right|^2}\right)\hat{S}_1(X)\,\delta\hat{f}(X,\theta-1)$$

$$\delta\hat{S}(X,\theta-1)\simeq\left(\frac{1}{2}-\frac{\frac{\partial\hat{K}_{X'}|\hat{\Psi}(X')|^2}{\partial\hat{f}(X,\theta-1)}}{\hat{K}_{X'}\left|\hat{\Psi}(X')\right|^2}\right)\hat{S}(X)\,\delta\hat{f}(X,\theta-1)$$

$$
\begin{aligned}
&\delta S_1(X,X,\theta-1)\\
\rightarrow\ & \frac{h(X)}{2}\left(\hat{h}(X)+\frac{h(X)}{2}\right)S_1(X,X)\,\delta f(X,\theta-1)\\
&+\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S_1(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\delta\hat{f}(X,\theta-1)+\frac{\partial_{f(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S_1(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\frac{\partial f(X)}{\partial S(X,\theta-2)}\delta S(X,\theta-2)\\
\rightarrow\ & \frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S_1(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\delta\hat{f}(X,\theta-1)\\
&+\left(\frac{h(X)}{2}\left(\hat{h}(X)+\frac{h(X)}{2}\right)S_1(X,X)+\frac{\partial_{f(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S_1(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\right)\frac{\partial f(X)}{\partial S(X,\theta-2)}\delta S(X,\theta-2)
\end{aligned}
$$

$$
\begin{aligned}
\delta S(X,\theta-1) &\simeq \frac{\hat{h}(X)}{2}\delta f(X,\theta-1)\,S(X)+\frac{\delta\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}S(X)\\
&\simeq \frac{\hat{h}(X)}{2}\delta f(X)\,S(X)-2S(X)\,\frac{\delta\hat{f}(X,\theta-1)}{\hat{f}(X,\theta-1)}-\frac{\delta f(X)}{f(X)+C_0}S(X)\\
&= \left(\frac{\hat{h}(X)}{2}-\frac{1}{f(X)+C_0}\right)S(X)\,\frac{\partial f(X)}{\partial S(X,\theta-2)}\delta S(X,\theta-2)-2S(X)\,\frac{\delta\hat{f}(X,\theta-1)}{\hat{f}(X,\theta-1)}
\end{aligned}
$$

We use in first approximation that:

$$
\begin{aligned}
\delta\frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)} &\simeq \delta\frac{1-\frac{1}{2}\hat{S}(X)}{1-\hat{S}(X)}\\
&= \frac{1}{2\left(1-\hat{S}(X)\right)^2}\delta\hat{S}(X)\\
&\simeq \frac{1}{2\left(1-\hat{S}(X)\right)^2}\left(\frac{1}{2}-\frac{\frac{\partial\hat{K}_{X'}\left|\hat{\Psi}(X')\right|^2}{\partial\hat{f}(X,\theta-1)}}{\hat{K}_{X'}\left|\hat{\Psi}(X')\right|^2}\right)\hat{S}(X)\,\delta\hat{f}(X,\theta-1)
\end{aligned}
$$

$$
\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\simeq-\frac{2S(X)}{\hat{f}(X,\theta-1)}
$$

$$
\delta\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\simeq-2\frac{\frac{\hat{h}(X)}{2}\delta f(X,\theta-1)\,S(X)-2S(X)\,\frac{\delta\hat{f}(X,\theta-1)}{\hat{f}(X,\theta-1)}-\frac{\frac{\partial f(X)}{\partial S(X,\theta-2)}}{f(X)+C_0}\delta S(X,\theta-2)}{\hat{f}(X,\theta-1)}
$$

$$\delta\alpha \simeq \left(\frac{\hat{S}(X)}{\left(1-\left(\hat{S}(X)\right)\right)}\left(\frac{1}{2\left(1+\frac{\Delta\hat{f}(X')+\Delta\hat{r}(X')}{2}\right)}-\frac{\frac{\partial\hat{K}_{X'}\left|\hat{\Psi}(X')\right|^2}{\partial\hat{f}(X,\theta-1)}}{\hat{K}_{X'}\left|\hat{\Psi}(X')\right|^2}\right)\right.$$
$$\left.-\frac{\hat{S}_1(X)}{\left(1-\left(\hat{S}_1(X')\right)\right)}\left(\frac{1}{1+\Delta\hat{f}(X)}-\frac{\frac{\partial\hat{K}_{X'}\left|\hat{\Psi}(X')\right|^2}{\partial\hat{f}(X,\theta-1)}}{\hat{K}_{X'}\left|\hat{\Psi}(X')\right|^2}\right)\right)\delta\hat{f}(X)$$
$$+\delta\left[\frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)}S_1(X,X,\theta-1)\frac{\partial f(X)}{\partial S(X,\theta-1)}\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\right] \tag{199}$$

In all expressions, due to presence of banks, $\hat{f}(X)$ stands for:

$$\hat{f}(X)+\frac{\left\langle\hat{S}(X',X)\right\rangle_{X'}\left\langle\hat{f}\right\rangle}{1-\left\langle\hat{S}\right\rangle}+\frac{\left(\left\langle\hat{S}_1^B(X,X)\right\rangle_X+\left\langle\hat{S}_2^B(X,X)\right\rangle_X+\left\langle\hat{S}(X',X)\right\rangle_{X'}\frac{\left\langle\hat{S}_1^B\right\rangle+\left\langle\hat{S}_2^B\right\rangle}{1-\left\langle\hat{S}\right\rangle}\right)\frac{\left\langle\bar{K}\right\rangle\left\|\bar{\Psi}\right\|^2}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\left\langle\bar{f}\right\rangle}{1-\left\langle\bar{S}\right\rangle}$$

We first derive:

$$\delta\left[\frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)}S_1(X,X,\theta-1)\frac{\partial f(X)}{\partial S(X,\theta-1)}\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\right]$$
$$\simeq \left[\frac{\hat{S}(X)}{2\left(1-\hat{S}(X)\right)^2}\left(\frac{1}{2}-\frac{\frac{\partial\hat{K}_{X'}\left|\hat{\Psi}(X')\right|^2}{\partial\hat{f}(X,\theta-1)}}{\hat{K}_{X'}\left|\hat{\Psi}(X')\right|^2}\right)S_1(X,X,\theta-1)\frac{\partial f(X)}{\partial S(X,\theta-1)}\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\right]\delta\hat{f}(X,\theta-1)$$
$$+\left[\frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)}S_1(X,X,\theta-1)\frac{\partial f(X)}{\partial S(X,\theta-1)}\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\right]$$
$$\times\left(\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\delta\hat{f}(X,\theta-1)+\left(\frac{h(X)}{2}\left(\hat{h}(X)+\frac{h(X)}{2}\right)+\frac{\partial_{f(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S_1(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\right)\right)$$
$$\times\frac{\partial f(X)}{\partial S(X,\theta-2)}\delta S(X,\theta-2)$$
$$-2\frac{\left(\frac{\hat{h}(X)}{2}S(X)-\frac{1}{f(X)+C_0}\right)\frac{\partial f(X)}{\partial S(X,\theta-2)}\delta S(X,\theta-2)-2S(X)\frac{\delta\hat{f}(X,\theta-1)}{\hat{f}(X,\theta-1)}}{\hat{f}(X,\theta-1)}$$
$$\times\left[\frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)}S_1(X,X,\theta-1)\frac{\partial f(X)}{\partial S(X,\theta-1)}\right]$$
$$+\left[\frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)}S_1(X,X,\theta-1)\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\right]\frac{\partial^2 f(X)}{\partial\left(S(X,\theta-2)\right)^2}\delta S(X,\theta-2)$$

and gather the terms to obtain $\delta\alpha$:

$$\delta\alpha \simeq T\delta\hat{f}(X)+\left(V\frac{\partial f(X)}{\partial S(X,\theta-2)}+W\frac{\partial^2 f(X)}{\partial\left(S(X,\theta-2)\right)^2}\right)\delta S(X,\theta-2) \tag{200}$$

$$
\begin{aligned}
T \;=\;& \left(\frac{\hat{S}(X)}{\left(1-\left(\hat{S}(X)\right)\right)}-\frac{\hat{S}_1(X)}{\left(1-\left(\hat{S}_1(X')\right)\right)}\right)\frac{2}{\hat{f}(X,\theta-1)}\\
&+\left[\frac{\hat{S}(X)}{\left(1-\hat{S}(X)\right)^2\hat{f}(X,\theta-1)}S_1(X,X,\theta-1)\frac{\partial f(X)}{\partial S(X,\theta-1)}\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\right]\\
&+\left[\frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)}S_1(X,X,\theta-1)\frac{\partial f(X)}{\partial S(X,\theta-1)}\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\right]\\
&\times\left(\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}+4\frac{1}{\left(\hat{f}(X,\theta-1)\right)^2\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}}\right)
\end{aligned}
$$

$$
\begin{aligned}
V =& \left[\frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)}S_1(X,X,\theta-1)\frac{\partial f(X)}{\partial S(X,\theta-1)}\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\right]\\
&\times\left(\frac{h(X)}{2}\left(\hat{h}(X)+\frac{h(X)}{2}\right)+\frac{\partial_{f(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S_1(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}-2\frac{\left(\frac{\hat{h}(X)}{2}S(X)-\frac{1}{f(X)+C_0}\right)}{\hat{f}(X,\theta-1)\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}}\right)
\end{aligned}
$$

$$
W=\left[\frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)}S_1(X,X,\theta-1)\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\right]
$$

$$
\begin{aligned}
\delta\beta \;\simeq\;& \left(4\frac{S(X)}{\left(\hat{f}(X,\theta-1)\right)^2}-\frac{\hat{h}(X)}{2}\frac{2}{\hat{f}(X,\theta-1)}S(X)\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)\delta\hat{f}(X,\theta-1)\\
&+\left(\frac{\hat{h}(X)}{2}S(X)-\frac{1}{f(X)+C_0}\right)\left(\frac{\hat{h}(X)}{2}-\frac{2}{\hat{f}(X,\theta-1)}\right)\left(\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)^2\delta S(X,\theta-2)+\beta\frac{\frac{\partial^2 f(X)}{\partial(S(X,\theta-2))^2}}{\frac{\partial f(X)}{\partial S(X,\theta-2)}}
\end{aligned}
$$

$$
\begin{aligned}
\delta c \simeq & \left\{\left(-\frac{\left(1-\left\langle \hat{S}(X')\right\rangle\right)}{1-\left\langle \hat{S}_1(X')\right\rangle}\frac{\left\langle \hat{h}(X',X)\right\rangle \langle h(X)\rangle}{4}-\frac{2S_1(X)\beta}{\hat{f}(X,\theta-1)}\right)\delta\left\langle \hat{f}(X')\right\rangle\right. \\
& +\left(\frac{h(X)}{4}\left(1+\hat{h}(X)\right)S_1(X)+S_1(X)\beta\left(\frac{\hat{h}(X)}{2}-\frac{1}{f(X)+C_0}\right)\right)\frac{\partial f(X)}{\partial S(X,\theta-2)}\delta S(X,\theta-2) \\
& \left.+S_1(X)\delta\beta\right\}\frac{\partial f(X)}{\partial S(X,\theta-2)}+c\frac{\frac{\partial^2 f(X)}{\partial (S(X,\theta-2))^2}}{\frac{\partial f(X)}{\partial S(X,\theta-2)}} \\
\simeq & S_1(X)\left(4\frac{S(X)}{\left(\hat{f}(X,\theta-1)\right)^2}-\frac{\hat{h}(X)}{2}\frac{2}{\hat{f}(X,\theta-1)}S(X)\frac{\partial f(X)}{\partial S(X,\theta-2)}\right. \\
& -\left(\frac{\left(1-\left\langle \hat{S}(X')\right\rangle\right)}{1-\left\langle \hat{S}_1(X')\right\rangle}\frac{\left\langle \hat{h}(X',X)\right\rangle \langle h(X)\rangle}{4}+\frac{2S_1(X)\beta}{\hat{f}(X,\theta-1)}\right)\Bigg)\frac{\partial f(X)}{\partial S(X,\theta-2)}\delta\left\langle \hat{f}(X')\right\rangle \\
& +\left(\left(\frac{\hat{h}(X)}{2}S(X)-\frac{1}{f(X)+C_0}\right)\left(\frac{\hat{h}(X)}{2}-\frac{2}{\hat{f}(X,\theta-1)}\right)\left(\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)^2\right. \\
& \left.+\left(\frac{h(X)}{4}\left(1+\hat{h}(X)\right)S_1(X)+S_1(X)\beta\left(\frac{\hat{h}(X)}{2}-\frac{1}{f(X)+C_0}\right)\right)\frac{\partial f(X)}{\partial S(X,\theta-2)}\right) \\
& \times\frac{\partial f(X)}{\partial S(X,\theta-2)}\delta S(X,\theta-2)+c\frac{\frac{\partial^2 f(X)}{\partial (S(X,\theta-2))^2}}{\frac{\partial f(X)}{\partial S(X,\theta-2)}}\delta S(X,\theta-2)
\end{aligned}
$$

$$
\delta h=4\frac{S(X)}{\left(\hat{f}(X,\theta-1)\right)^2}\delta\hat{f}(X,\theta-1)-2\frac{\left(\frac{\hat{h}(X)}{2}S(X)-\frac{1}{f(X)+C_0}\right)}{\hat{f}(X,\theta-1)}\frac{\partial f(X)}{\partial S(X,\theta-2)}\delta S(X,\theta-2)
$$

A4.4.2 Dominant contributions

The dominant contributions in $\frac{\partial f(X)}{\partial S(X,\theta-1)}$ and $\frac{1}{\left(\hat{f}(X,\theta-1)\right)} << 1$:

$$
\begin{aligned}
T \simeq & \left[\frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)}S_1(X,X)\frac{\partial f(X)}{\partial S(X,\theta-1)}\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\right] \\
& \times\left(\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}+4\frac{1}{\left(\hat{f}(X,\theta-1)\right)^2\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}}\right) \\
\simeq & \frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)}S_1(X,X)\frac{\partial f(X)}{\partial S(X,\theta-1)}\frac{8}{\left(\hat{f}(X,\theta-1)\right)^2}
\end{aligned}
$$

$$\begin{aligned}
\delta\alpha &\simeq S_1(X,X)\frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)}\frac{8}{\left(\hat{f}(X,\theta-1)\right)^2}\frac{\partial f(X)}{\partial S(X,\theta-1)}\delta\hat{f}(X) \\
&+\left[\frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)}S_1(X,X)\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\right] \\
&\times\left(\frac{h(X)}{2}\left(\hat{h}(X)+\frac{h(X)}{2}\right)+\frac{\partial_{f(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S_1(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}-2\frac{\left(\frac{\hat{h}(X)}{2}S(X)-\frac{1}{f(X)+C_0}\right)}{\hat{f}(X,\theta-1)\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}}\right) \\
&\times\left(\left(\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)^2+\frac{\partial^2 f(X)}{\partial\left(S(X,\theta-2)\right)^2}\right)\delta S(X,\theta-2)
\end{aligned} \tag{201}$$

$$\begin{aligned}
\delta\beta &\simeq S(X)\left(\frac{4}{\left(\hat{f}(X,\theta-1)\right)^2}-\frac{\hat{h}(X)}{\hat{f}(X,\theta-1)}\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)\delta\hat{f}(X,\theta-1) \\
&-\frac{2}{\hat{f}(X,\theta-1)}\left(\frac{\hat{h}(X)}{2}S(X)-\frac{1}{f(X)+C_0}\right)\left(\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)^2\delta S(X)+\beta\frac{\frac{\partial^2 f(X)}{\partial(S(X,\theta-2))^2}}{\frac{\partial f(X)}{\partial S(X,\theta-2)}}\delta S(X)
\end{aligned}$$

$$\begin{aligned}
\delta c &\simeq -\frac{2S_1(X)\beta}{\hat{f}(X,\theta-1)}\delta\left\langle\hat{f}(X')\right\rangle \\
&+\left(\frac{\hat{h}(X)}{2}S(X)-\frac{1}{f(X)+C_0}\right)\left(\frac{\hat{h}(X)}{2}-\frac{2}{\hat{f}(X,\theta-1)}\right)\left(\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)^3\delta S(X)+c\frac{\frac{\partial^2 f(X)}{\partial(S(X,\theta-2))^2}}{\frac{\partial f(X)}{\partial S(X,\theta-2)}}\delta S(X) \\
&-\left(\hat{h}(X)+2\left(\frac{\hat{h}(X)}{2}+\frac{\partial_{f(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\right)\right)\frac{S_1(X)S(X)}{\hat{f}(X,\theta-1)}\left(\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)^2\delta\left\langle\hat{f}(X')\right\rangle
\end{aligned}$$

In general, due to bank loans:

$$S_1(X,X)S(X)<<1$$

so that:

$$\begin{aligned}
\delta c &\simeq -\frac{2S_1(X)\beta}{\hat{f}(X,\theta-1)}\delta\left\langle\hat{f}(X')\right\rangle \\
&+\left(\frac{\hat{h}(X)}{2}S(X)-\frac{1}{f(X)+C_0}\right)\left(\frac{\hat{h}(X)}{2}-\frac{2}{\hat{f}(X,\theta-1)}\right)\left(\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)^3\delta S(X)+c\frac{\frac{\partial^2 f(X)}{\partial(S(X,\theta-2))^2}}{\frac{\partial f(X)}{\partial S(X,\theta-2)}}\delta S(X) \\
&\rightarrow -\frac{2S_1(X)\beta}{\hat{f}(X,\theta-1)}\delta\left\langle\hat{f}(X')\right\rangle-\frac{2}{\hat{f}(X,\theta-1)}\left(\frac{\hat{h}(X)}{2}S(X)-\frac{1}{f(X)+C_0}\right)\left(\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)^3\delta S(X)
\end{aligned}$$

$$\delta h \simeq 4\frac{S(X)}{\left(\hat{f}(X,\theta-1)\right)^2}\delta\hat{f}(X,\theta-1) - 2\frac{\left(\frac{\hat{h}(X)}{2}S(X) - \frac{1}{f(X)+C_0}\right)}{\hat{f}(X,\theta-1)}\frac{\partial f(X)}{\partial S(X,\theta-2)}\delta S(X,\theta-2)$$

$$\begin{aligned}\delta\beta \quad &\simeq \quad \left(4\frac{S(X)}{\left(\hat{f}(X,\theta-1)\right)^2} - \frac{\hat{h}(X)}{2}\frac{2}{\hat{f}(X,\theta-1)}S(X)\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)\delta\hat{f}(X,\theta-1) \\ &\quad -\left(\frac{\hat{h}(X)}{2}S(X) - \frac{1}{f(X)+C_0}\right)\frac{2}{\hat{f}(X,\theta-1)}\left(\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)^2\delta S(X,\theta-2)\end{aligned}$$

A4.4.3 Stability condition

As for banks, we can derive a stabilit condtion:

$$\begin{aligned}&\left(1-\langle\lambda^2\rangle\right)\left((\delta S(X))^2 + \left(\delta\hat{S}(X)\right)^2\right) \\ > \quad &\left\{\frac{\delta\alpha}{\delta\hat{f}(X)}\left(\delta\hat{S}(X)\right)^2 + \delta\hat{S}(X)\left(\frac{\delta\alpha}{\delta S(X)} + \frac{\hat{h}(X',X)}{2}\frac{\delta c}{\delta\hat{f}(X)}\right)\delta S(X) + \frac{\hat{h}(X',X)}{2}\frac{\delta c}{\delta S(X)}(\delta S(X))^2\right\}^2 \\ &+\left\{\frac{1}{\frac{\hat{h}(X',X)}{2}}\frac{\delta h}{\delta\hat{f}(X,\theta-1)}\left(\delta\hat{S}(X)\right)^2\right. \\ &\left.+\delta\hat{S}(X)\left(\frac{1}{\frac{\hat{h}(X',X)}{2}}\frac{\delta}{\delta S(X)}h + \frac{\delta\beta}{\delta\hat{f}(X,\theta-1)}\right)\delta S(X) + \frac{\delta\beta}{\delta S(X,,\theta-1)}(\delta S(X))^2\right\}^2\end{aligned}$$

For $\hat{f}(X,\theta-1) << 1$, $\left|\frac{\partial f(X)}{\partial S(X,\theta-2)}\right| > 1$ we use that:

$$\frac{\delta\alpha}{\delta S(X)} + \frac{\hat{h}(X',X)}{2}\frac{\delta c}{\delta\hat{f}(X)} \simeq \frac{\delta\alpha}{\delta S(X)}$$

is of order:

$$\frac{\left(\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)^2 + \frac{\partial^2 f(X)}{\partial(S(X,\theta-2))^2}}{\hat{f}(X,\theta-1)}$$

is negligible with respect to:

$$\sqrt{\left|\frac{\delta\alpha}{\delta\hat{f}(X)}\frac{\delta c}{\delta S(X)}\right|}$$

which is of order:

$$\sqrt{\frac{1}{\left(\hat{f}(X,\theta-1)\right)^3}\left(\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)^4}$$

Moreover:

$$\begin{aligned}&\left|\frac{\delta}{\delta S(X)}h\right| \\ \rightarrow \quad &2\frac{\left|\frac{\hat{h}(X)}{2}S(X) - \frac{1}{f(X)+C_0}\right|}{\hat{f}(X,\theta-1)}\frac{\partial f(X)}{\partial S(X,\theta-2)}\end{aligned}$$

$$\left|\frac{\delta\beta}{\delta\hat{f}(X,\theta-1)}\right| \simeq S(X)\left|4\frac{1}{\left(\hat{f}(X,\theta-1)\right)^2} - \frac{\hat{h}(X)}{2}\frac{2}{\hat{f}(X,\theta-1)}\frac{\partial f(X)}{\partial S(X,\theta-2)}\right|$$

and:

$$\left|\frac{\delta h}{\delta\hat{f}(X,\theta-1)}\frac{\delta\beta}{\delta S(X,,\theta-1)}\right|$$
$$\rightarrow\quad 4\frac{S(X)}{\left(\hat{f}(X,\theta-1)\right)^3}\left(\left|\frac{\hat{h}(X)}{2}S(X) - \frac{1}{f(X)+C_0}\right|\right)\left(\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)^2$$

For:

$$\hat{f}(X,\theta-1) << 1, \left|\frac{\partial f(X)}{\partial S(X,\theta-2)}\right| > 1, S(X) << 1, \frac{1}{f(X)+C_0} << 1$$

we have in most cases:

$$\frac{\sqrt{S(X)}}{\hat{f}(X,\theta-1)} << 1$$

$$\left|\frac{\delta}{\delta S(X)}h\right| < \sqrt{\left|\frac{\delta h}{\delta\hat{f}(X,\theta-1)}\frac{\delta\beta}{\delta S(X,,\theta-1)}\right|}$$
$$\left|\frac{\delta\beta}{\delta\hat{f}(X,\theta-1)}\right| < \sqrt{\left|\frac{\delta h}{\delta\hat{f}(X,\theta-1)}\frac{\delta\beta}{\delta S(X,,\theta-1)}\right|}$$

As a consequence, the condition approximates:

$$\left(1-\langle\lambda^2\rangle\right)\left((\delta S(X))^2 + \left(\delta\hat{S}(X)\right)^2\right)$$
$$> \left\{\frac{\delta\alpha}{\delta\hat{f}(X)}\left(\delta\hat{S}(X)\right)^2 + \frac{\hat{h}(X',X)}{2}\frac{\delta c}{\delta S(X)}(\delta S(X))^2\right\}^2$$
$$+\left\{\frac{1}{\frac{\hat{h}(X',X)}{2}}\frac{\delta h}{\delta\hat{f}(X,\theta-1)}\left(\delta\hat{S}(X)\right)^2 + \frac{\delta\beta}{\delta S(X,,\theta-1)}(\delta S(X))^2\right\}^2$$

that is:

$$\left(1-\langle\lambda^2\rangle\right)\left((\delta S(X))^2 + \left(\delta\hat{S}(X)\right)^2\right)$$
$$> \left\{\frac{8S_1(X,X)\frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)}}{\left(\hat{f}(X,\theta-1)\right)^2}\frac{\partial f(X)}{\partial S(X,\theta-1)}\left(\delta\hat{S}(X)\right)^2\right.$$
$$\left.-\frac{\hat{h}(X',X)}{2}\frac{2\left(\frac{\hat{h}(X)}{2}S(X) - \frac{1}{f(X)+C_0}\right)\left(\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)^3(\delta S(X))^2}{\hat{f}(X,\theta-1)}\right\}^2$$
$$+\left\{\frac{1}{\frac{\hat{h}(X',X)}{2}}4\frac{S(X)\left(\delta\hat{S}(X)\right)^2}{\left(\hat{f}(X,\theta-1)\right)^2} - \frac{2\left(\frac{\hat{h}(X)}{2}S(X) - \frac{1}{f(X)+C_0}\right)}{\hat{f}(X,\theta-1)}\left(\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)^2(\delta S(X))^2\right\}^2$$

Due to bank loans $S_1(X,X) << 1$ and for $\left|\frac{\partial f(X)}{\partial S(X,\theta-2)}\right| > 1$ this reduces to:

$$1-\left\langle \lambda^2 \right\rangle > \frac{\left\{ \frac{8S_1(X,X)\frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)}}{\left(\hat{f}(X,\theta-1)\right)^2} \frac{\partial f(X)}{\partial S(X,\theta-1)} \left(\delta \hat{S}(X)\right)^2 - \frac{\hat{h}(X',X)}{2} \frac{2\left(\frac{\hat{h}(X)}{2}S(X)-\frac{1}{f(X)+C_0}\right)}{\hat{f}(X,\theta-1)} \left(\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)^3 (\delta S(X))^2 \right\}^2}{(\delta S(X))^2 + \left(\delta \hat{S}(X)\right)^2}$$

The estimation of $\langle \lambda^2 \rangle$ comes from:

$$\alpha = \frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)} S_1(X,X) \frac{\partial f(X)}{\partial S(X)} \frac{\partial_{\hat{f}(X)} \left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right) S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}} \tag{202}$$

$$\begin{aligned}
\beta &\simeq \left(\frac{\hat{h}(X)}{2}S(X) + \frac{\partial_{f(X)} \left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right) S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\right) \frac{\partial f(X)}{\partial S(X)} \\
&\to \left(\frac{\hat{h}(X)}{2}S(X) + \frac{1}{f(X)+C_0}\right) \frac{\partial f(X)}{\partial S(X)}
\end{aligned}$$

$$c \simeq S_1(X) \left(\frac{\hat{h}(X)}{2}S(X) + \frac{\partial_{f(X)} \left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right) S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\right) \left(\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)^2$$

$$\begin{aligned}
h &= \frac{\partial_{\hat{f}(X)} \left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right) S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}} \\
&\to -2\frac{S(X)}{\hat{f}(X)}
\end{aligned}$$

$$\begin{aligned}
\langle \lambda \rangle &= \frac{1}{2}\alpha + \frac{1}{2}\beta \\
&= \frac{1}{2}\left(\left(\frac{\hat{h}(X)}{2} + \frac{1}{f(X)+C_0}\right) - \frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)} S_1(X,X) \frac{2}{\hat{f}(X)}\right) S(X) \frac{\partial f(X)}{\partial S(X,\theta-2)} \\
&\to -\frac{2S_1(X,X)S(X)\left(1-\left(\hat{S}_1(X)\right)\right)}{\hat{f}(X)\left(1-\left(\hat{S}(X)\right)\right)} \frac{\partial f(X)}{\partial S(X,\theta-2)} > 0
\end{aligned}$$

and the condition writes:

$$1-\left(\frac{2S_1\left(X,X\right)S\left(X\right)\left(1-\left(\hat{S}_1\left(X\right)\right)\right)}{\hat{f}\left(X\right)\left(1-\left(\hat{S}\left(X\right)\right)\right)}\frac{\partial f\left(X\right)}{\partial S\left(X,\theta-2\right)}\right)^2$$

$$>\frac{\left\{\frac{8S_1(X,X)\frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)}}{\left(\hat{f}(X,\theta-1)\right)^2}\left(\delta\hat{S}\left(X\right)\right)^2-\frac{\hat{h}\left(X',X\right)}{2}\frac{2\left(\frac{\hat{h}(X)}{2}S(X)-\frac{1}{f(X)+C_0}\right)}{\hat{f}(X,\theta-1)}\left(\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)^2\left(\delta S\left(X\right)\right)^2\right\}^2\left(\frac{\partial f(X)}{\partial S(X,\theta-1)}\right)^2}{\left(\delta S\left(X\right)\right)^2+\left(\delta\hat{S}\left(X\right)\right)^2}$$

Appendix 5 Fiber stability

A5.1 Banks

A5.1.1 Fiber fluctuations

The effect of fluctuations in investors stakes as been studied in Gosselin and Lotz 25 a and we focus on banks only. To evaluate the effect of introducing banks, we consider:

$$\begin{array}{rcl}
S_L^B\left(X,\theta-1\right) & > & >S_E^B\left(X,\theta-1\right)\\
S_L^B\left(X,\theta-1\right) & > & >S\left(X\right)\\
 & & \hat{S}\left(X\right)
\end{array}$$

$$\begin{array}{rcl}
a & > & >b\\
d & > & >e\\
h & > & >
\end{array}$$

so that in first approximation we replace the three dimensional system:

$$\left(\begin{array}{c}\delta\bar{f}\left(X,\theta\right)\\ \delta\hat{f}\left(X,\theta\right)\\ \delta S^T\left(X,,\theta-1\right)\end{array}\right)=\left[\begin{array}{ccc}a & 0 & c\\ d & -1 & f\\ g & 0 & i\end{array}\right]\left(\begin{array}{c}\delta\bar{f}\left(X,\theta-1\right)\\ \delta\hat{f}\left(X,\theta-1\right)\\ \delta S^T\left(X,,\theta-2\right)\end{array}\right)$$

by:

$$X\left(\theta\right)=MX\left(\theta-1\right)$$

with:

$$X\left(\theta\right)=\left(\begin{array}{cccccc}\delta\bar{f}\left(X,\theta\right) & \delta\hat{f}\left(X,\theta\right) & \delta S^T\left(X,,\theta-1\right) & \frac{\delta\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2} & \frac{\delta\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{\hat{K}_X\left|\hat{\Psi}(X)\right|^2} & \frac{\delta K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\end{array}\right)^t$$

The matrix M is found by writing the various equatios. Their are similar to the ones obtained above, but take into account that the disposable capitals for banks, investors and firms are now independent.

We write:

$$\frac{1-\bar{S}(X)}{1-\bar{S}_E(X)}\delta\bar{f}(X,\theta) = a\delta\bar{f}(X,\theta-1)+a_0\frac{\delta\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}}{\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}}+a_1\frac{\delta\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}$$

$$+b\frac{\delta\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\right)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}}+c\delta S^T(X,\theta-2)$$

$$a = \frac{1}{2}H(X)\bar{S}(X)\frac{\left(\bar{f}(X)-\bar{r}\right)}{1-\left(\bar{S}_E(X',\theta-1)\right)}$$

$$-\left(H(X)\bar{S}_E(X)\right)\frac{\left(1-\left(\bar{S}(X,\theta-1)\right)\right)\left(\bar{f}(X)-\bar{r}\right)}{\left(1-\left(\bar{S}_E(X',\theta-1)\right)\right)^2}$$

$$a_0=\left(S_E^B(X,\theta-1)+S_L^B(X,\theta-1)\right)S_E^B(X,X)$$

$$a_1 = -\bar{S}(X',\theta-1)\frac{\left(\bar{f}(X)-\bar{r}\right)}{1-\left(\bar{S}_E(X',\theta-1)\right)}$$

$$+\bar{S}_E(X',\theta-1)\frac{\left(1-\left(\bar{S}(X,\theta-1)\right)\right)\left(\bar{f}(X)-\bar{r}\right)}{\left(1-\left(\bar{S}_E(X',\theta-1)\right)\right)^2}$$

$$b=S_E^B(X,X)S(X)$$

$$c = \left\{w_1^B(X)\left(\langle\bar{w}(X)\rangle+\left\langle\hat{w}_1^B(X)\right\rangle\right)(f(X)-\bar{r})\frac{\partial f(X)}{\partial S^T(X,\theta-2)}-\frac{\langle\bar{w}(X',X)\rangle}{2}w_1^B(X)\frac{1-\left\langle\bar{S}(X')\right\rangle}{1-\left\langle\bar{S}_E(X')\right\rangle}\left(\left\langle\bar{f}(X')\right\rangle-\bar{r}\right)\right.$$

$$-\left\langle\hat{w}_1^B(X',X)\right\rangle\left\langle w_1^B(X)\right\rangle\frac{1-\left\langle\hat{S}(X')\right\rangle+\left\langle\hat{S}_E^B(X')\right\rangle+\left\langle\hat{S}_L^B(X')\right\rangle}{1-\left\langle\hat{S}_E(X')\right\rangle+\left\langle\hat{S}_E^B(X')\right\rangle}\left(\left\langle\hat{f}(X')\right\rangle-\bar{r}\right)$$

$$\left.+\frac{S_E^B(X,X)\partial f(X)}{\partial S^T(X,\theta-1)}\left\{w_1^B(X)\left(\langle\bar{w}(X)\rangle+\left\langle\hat{w}_1^B(X)\right\rangle\right)+\frac{\frac{\hat{w}(X)}{2}S(X)}{1+\left(\hat{w}(X)\left(\frac{f(X)+\bar{r}(X)}{2}-\frac{\left\langle\hat{f}(X')\right\rangle_{\hat{w}_1}+\langle\hat{r}(X')\rangle_{\hat{w}_2}}{2}\right)\right)}\right\}\right\}$$

$$\times\frac{\partial f(X)}{\partial S^T(X,\theta-2)}$$

$$\frac{1-\left(\hat{S}(X)\right)}{1-\left(\hat{S}_E(X)\right)}\delta\hat{f}(X,\theta)=d\frac{\delta\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}}{\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}}+e\delta\hat{f}(X,\theta-1)+e_0\frac{\delta\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}+f\delta S^T(X,\theta-2)$$

$$d=S_E(X,X,\theta-1)\left(S_E^B(X,\theta-1)+S_L^B(X,\theta-1)\right)$$

$$e = \left(\frac{\hat{S}(X)}{2\left(1-\left(\hat{S}_E(X')\right)\right)}\frac{1}{1+\frac{\Delta\hat{f}(X')+\Delta\hat{r}(X')}{2}}\right.$$

$$\left.-\frac{\left(1-\left(\hat{S}(X)\right)\right)\hat{S}_E(X)}{\left(1-\left(\hat{S}_E(X')\right)\right)^2}\left(\frac{1}{1+\Delta\hat{f}(X)}\right)\right)\left(\hat{f}(X)-\bar{r}\right)$$

$$e_0 = \left(\frac{\left(1 - \left(\hat{S}(X)\right)\right) \hat{S}_E(X)}{\left(1 - \left(\hat{S}_E(X')\right)\right)^2} - \frac{\hat{S}(X)}{2\left(1 - \left(\hat{S}_E(X')\right)\right)} \right) \left(\hat{f}(X) - \bar{r}\right) + S_E(X, X, \theta - 1)$$

$$\begin{aligned}
f = & \left\{ - \frac{\left(1 - \left\langle \hat{S}(X') \right\rangle\right) \left(\left\langle \hat{f}(X') \right\rangle - \bar{r} \right)}{1 - \left\langle \hat{S}_E(X') \right\rangle} \frac{\langle \hat{w}(X', X) \rangle \langle w(X) \rangle}{4} \right. \\
& + \frac{\frac{w(X)}{4} (1 + \hat{w}(X)) S_E(X)}{1 + \left(\hat{w}(X) \left(f(X) - \frac{\left\langle \hat{f}(X') \right\rangle_{\hat{w}_1} + \langle \hat{r}(X') \rangle_{\hat{w}_2}}{2} \right) + \frac{w(X)}{2} (f(X) - \bar{r}(X)) \right)} (f(X) - \bar{r}) \\
& + S_E(X, X, \theta - 1) \frac{\partial f(X)}{\partial S^T(X, \theta - 1)} \\
& \left. \times \left(w_1^B(X) \left(\langle \bar{w}(X) \rangle + \left\langle \hat{w}_1^B(X) \right\rangle \right) + \frac{\frac{\hat{w}(X)}{2} S(X)}{1 + \left(\hat{w}(X) \left(\frac{f(X) + \bar{r}(X)}{2} - \frac{\left\langle \hat{f}(X') \right\rangle_{\hat{w}_1} + \langle \hat{r}(X') \rangle_{\hat{w}_2}}{2} \right) \right)} \right) \right\} \\
& \times \frac{\partial f(X)}{\partial S^T(X, \theta - 2)}
\end{aligned}$$

$$\delta S^T(X, , \theta - 1) = g \frac{\delta \frac{\bar{K}_X \left|\bar{\Psi}(X)\right|^2}{K_X |\Psi(X)|^2}}{\frac{\bar{K}_X \left|\bar{\Psi}(X)\right|^2}{K_X |\Psi(X)|^2}} + h \frac{\delta \left(\frac{\hat{K}_X \left|\hat{\Psi}(X)\right|^2}{K_X |\Psi(X)|^2} \right)}{\frac{\hat{K}_X \left|\hat{\Psi}(X)\right|^2}{K_X |\Psi(X)|^2}} + i \delta S^T(X, \theta - 2)$$

$$\begin{aligned}
g &= \left(S_E^B(X, \theta - 1) + S_L^B(X, \theta - 1) \right) \\
h &= S(X) \\
i &= \left\{ w_1^B(X) \left(\langle \bar{w}(X) \rangle + \left\langle \hat{w}_1^B(X) \right\rangle \right) \right. \\
& \quad \left. + \left(\frac{\frac{\hat{w}(X)}{2} S(X)}{1 + \left(\hat{w}(X) \left(\frac{f(X) + \bar{r}(X)}{2} - \frac{\left\langle \hat{f}(X') \right\rangle_{\hat{w}_1} + \langle \hat{r}(X') \rangle_{\hat{w}_2}}{2} \right) \right)} \right) \right\} \frac{\partial f(X)}{\partial S^T(X, \theta - 2)}
\end{aligned}$$

As before, we focus on banks only and consider the case of investors independently. To evaluate the effect of introducing banks, we consider:

$$\begin{aligned}
S_L^B(X, \theta - 1) &> > S_E^B(X, \theta - 1) \\
S_L^B(X, \theta - 1) &> > S(X) \\
& \hat{S}(X)
\end{aligned}$$

$$\begin{aligned}
a &> > b \\
d &> > e \\
h &> >
\end{aligned}$$

so that in first approximation the system is:

$$X(\theta) = M X(\theta - 1)$$

with:

$$M=\begin{bmatrix} a & 0 & c & a_0+a_1 & b & -a_0-b \\ 0 & -1 & f & d & e_0 & -d \\ 0 & 0 & i & g & h & -g-h \\ \frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2} & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial_{\bar{f}(X)}\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{\hat{K}_X\left|\hat{\Psi}(X)\right|^2} & \frac{\partial_{\hat{f}(X)}\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{\hat{K}_X\left|\hat{\Psi}(X)\right|^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\frac{\partial f(X)}{\partial S^T(X,\theta-2)} & 0 & 0 & 0 \end{bmatrix}$$

13.1.1 A5.1.2 Eigenvalues

As befr we assm that $\left|\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right| >> 1$. We set $z=\frac{\partial f(X)}{\partial S^T(X,\theta-1)}$ and rescale the matrix tht writes in first approximation:

$$M=\begin{pmatrix} 0 & 0 & cz^2 & a_0 & b & -a-b \\ 0 & -1 & fz^2 & d & e_0 & -d \\ 0 & 0 & iz & g & h & -g-h \\ r & 0 & 0 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 0 & 0 \\ 0 & 0 & tz & 0 & 0 & 0 \end{pmatrix}$$

with:

$$r=\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}\simeq-\frac{2}{\left(1-\left\langle\bar{S}\right\rangle\right)\bar{f}(X)+\left\langle\bar{S}(X',X)\right\rangle_{X'}\left\langle\bar{f}\right\rangle}<0$$

$$s=\frac{\partial_{\hat{f}(X)}\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}\simeq-\frac{2}{\hat{f}(X)+\frac{\left(\left\langle\hat{S}_E^B(X,X')\right\rangle_{X'}+\left\langle\hat{S}_L^B(X,X')\right\rangle_{X'}\right)\frac{\langle\bar{K}\rangle\|\bar{\Psi}\|^2}{\langle\hat{K}\rangle\|\hat{\Psi}\|^2}\left\langle\bar{f}\right\rangle}{1-\left\langle\bar{S}\right\rangle}}<0$$

$$\left|\frac{\partial_{\bar{f}(X)}\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}\right|\simeq\frac{2\frac{\left(\left\langle\hat{S}_E^B(X,X')\right\rangle_{X'}+\left\langle\hat{S}_L^B(X,X')\right\rangle_{X'}\right)\frac{\langle\bar{K}\rangle\|\bar{\Psi}\|^2}{\langle\hat{K}\rangle\|\hat{\Psi}\|^2}\left\langle\bar{f}\right\rangle}{1-\left\langle\bar{S}\right\rangle}}{\hat{f}(X)+\frac{\left(\left\langle\hat{S}_E^B(X,X')\right\rangle_{X'}+\left\langle\hat{S}_L^B(X,X')\right\rangle_{X'}\right)\frac{\langle\bar{K}\rangle\|\bar{\Psi}\|^2}{\langle\hat{K}\rangle\|\hat{\Psi}\|^2}\left\langle\bar{f}\right\rangle}{1-\left\langle\bar{S}\right\rangle}}<<s$$

$$t=\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\simeq\frac{2}{r}\frac{1}{f(X)+C_0}>0$$

At the lowest order the eigenvalue satisfy:

$$X\left(X^2(cgr+fhs)+Xcgr-cgrse-afhrs+bfgrs+cdhrs\right)=0$$

$$X^2+X\frac{cgr}{(cgr+fhs)}-\frac{rs}{(cgr+fhs)}(cge+afh-bfg-cdh)$$

where:

$$\frac{cgr}{(cgr+fhs)}>0$$

and:

$$\begin{aligned}
& (cge + afh - bfg - cdh) \\
= \; & c(ge - dh) + f(ah - bg) \\
= \; & S_E(X, X, \theta - 1)\left(S_E^B(X, \theta - 1) + S_L^B(X, \theta - 1)\right)(1 - S(X)) \\
& + \left(S_E^B(X, \theta - 1) + S_L^B(X, \theta - 1)\right) S_E^B(X, X) S(X) - S_E^B(X, X) S(X)\left(S_E^B(X, \theta - 1) + S_L^B(X, \theta - 1)\right) \\
= \; & S_E(X, X, \theta - 1)\left(S_E^B(X, \theta - 1) + S_L^B(X, \theta - 1)\right)(1 - S(X)) > 0
\end{aligned}$$

then:

$$-\frac{rs}{(cgr + fhs)}(cge + afh - bfg - cdh) > 0$$

Eigenvalues are negative.

To obtain the eigenvalues close to 0, we set $X = \frac{\epsilon}{z}$ and the lowest order equation becomes:

$$\epsilon^2 + \left(z - iz^2\right)\epsilon - \left(sz^2 e + arz^2\right) = 0$$

Given that:

$$-\left(sz^2 e + arz^2\right) > 0$$

and $i < 0$:

$$\left(z - iz^2\right) > 0$$

Thus the eigenvalues are negative in general. This implies stability in main cases.

A5.1.3 Range of stability

A5.1.3.1 Estimation of the coefficients As before we study the range os stability by considering the second order system:

$$X(\theta) = MX(\theta - 1) + X^t(\theta - 1)\mathbf{M}X(\theta - 1)$$

with:

$$\mathbf{M} = \begin{pmatrix} \bar{M} & \hat{M} & M^T & M_{\bar{K}} & M_{\hat{K}} & M_K \end{pmatrix}$$

where the matrices are defined below.

We consider that:

$$S_1^B(X, \theta - 1) + S_2^B(X, \theta - 1) >> 1$$

due to bank loans. We consider also

$$\frac{\partial_{\bar{f}(X)} \bar{K}_X \left|\bar{\Psi}(X)\right|^2}{\bar{K}_X \left|\bar{\Psi}(X)\right|^2} << 1$$

and:

$$\frac{\partial_{f(X)} K_X \left|\Psi(X)\right|^2}{K_X \left|\Psi(X)\right|^2} \simeq \frac{2}{r}\frac{1}{f(X) + C_0}$$

Moreover, we use:

$$\frac{\hat{K}_X \left|\hat{\Psi}(X)\right|^2}{K_X \left|\Psi(X)\right|^2} \simeq \frac{18\sigma_{\hat{K}}^2 V \left\|\bar{\Psi}_0(X)\right\|^4}{\hat{\mu}F_1^2\left(\left(\frac{2\epsilon}{3\sigma_{\hat{K}}^2}\right)^{\frac{r}{2}} \frac{f_1(X)}{C_0 + \frac{S_2(X)}{1 - S_1(X)}\bar{r}}\right)^{\frac{2}{r}}}$$

and thus:

$$\frac{\partial_{f(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}} \simeq -\frac{2}{r}\frac{1}{f(X)+C_0}$$

$$\frac{\delta}{\delta S^T(X,,\theta-2)}\frac{\partial_{f(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}} \simeq \frac{2}{r}\frac{1}{(f(X)+C_0)^2}\frac{\delta f(X)}{\delta S^T(X,,\theta-2)}$$

$$\frac{\delta}{\delta S^T(X,,\theta-2)}\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2} \simeq -\frac{2}{r}\frac{1}{(f(X)+C_0)^2}\frac{\delta f(X)}{\delta S^T(X,,\theta-2)}$$

$$\begin{aligned}
&\frac{\partial}{\partial S^T(X,\theta-2)}\left(S_1^B(X,X)\,h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)\right)\\
=\;& h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)\\
&\times\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)\right)\\
&-S_1^B(X,X)\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)\right)
\end{aligned}$$

$$\begin{aligned}
&\frac{\delta}{\delta S^T(X,,\theta-2)}\left(S_1^B(X,X)\left(S_1^B(X,\theta-1)+S_2^B(X,\theta-1)\right)\right)\\
=\;& S_1^B(X,X)\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)\right)\\
&\times\frac{\partial f(X)}{\partial S^T(X,\theta-2)}+\left(S_1^B(X,\theta-1)+S_2^B(X,\theta-1)\right)h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)\frac{\partial f(X)}{\partial S^T(X,\theta-2)}
\end{aligned}$$

The derivatives are obtained as:

$$\begin{aligned}
&\frac{\delta}{\delta\bar{f}(X,\theta-1)}a\\
\simeq\;&\left(\frac{1}{2}H(X)\bar{S}(X)\right)\frac{1}{1-\left(\bar{S}_1(X',\theta-1)\right)}-\left(H(X)\bar{S}_1(X)\right)\frac{\left(1-\left(\bar{S}(X,\theta-1)\right)\right)}{\left(1-\left(\bar{S}_1(X',\theta-1)\right)\right)^2}
\end{aligned}$$

$$\frac{\delta}{\delta\bar{f}(X,\theta-1)}a_0=\left(\left(2S_1^B(X,\theta-1)+S_2^B(X,\theta-1)\right)h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)\right)\frac{\partial f(X)}{\partial S^T(X,\theta-2)}$$

$$\frac{\delta}{\delta\bar{f}(X,\theta-1)}a_1=-\bar{S}(X',\theta-1)\frac{1}{1-\left(\bar{S}_E(X',\theta-1)\right)}+\bar{S}_E(X',\theta-1)\frac{1-\left(\bar{S}(X,\theta-1)\right)}{\left(1-\left(\bar{S}_E(X',\theta-1)\right)\right)^2}$$

$$\frac{\delta}{\delta\bar{f}(X,\theta-1)}b=0$$

$$\frac{\delta}{\delta\hat{f}(X,\theta-1)}a\simeq\frac{\delta}{\delta\hat{f}(X,\theta-1)}a_0\simeq\frac{\delta}{\delta\hat{f}(X,\theta-1)}a_1\simeq\frac{\delta}{\delta\hat{f}(X,\theta-1)}b\simeq0$$

$$\frac{\delta}{\delta S^T(X,\theta-2)}a\simeq\frac{\delta}{\delta S^T(X,\theta-2)}a_0\simeq\frac{\delta}{\delta S^T(X,\theta-2)}a_1\simeq\frac{\delta}{\delta S^T(X,\theta-2)}b\simeq0$$

$$\frac{\delta}{\delta\bar{f}(X,\theta-1)}c = \left\{h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)\frac{\partial f(X)}{\partial S^T(X,\theta-2)}-\frac{\left\langle\bar{h}(X',X)\right\rangle}{2}h_1^B(X)\frac{1-\left\langle\bar{S}(X')\right\rangle}{1-\left\langle\bar{S}_1(X')\right\rangle}\right\}$$
$$\times\frac{\partial f(X)}{\partial S^T(X,\theta-2)}$$

$$\frac{\delta}{\delta\hat{f}(X,\theta-1)}c\simeq-\left\langle\hat{h}_1^B(X',X)\right\rangle\left\langle h_1^B(X)\right\rangle\frac{1-\left\langle\hat{S}(X')\right\rangle+\left\langle\hat{S}_1^B(X')\right\rangle+\left\langle\hat{S}_2^B(X')\right\rangle}{1-\left\langle\hat{S}_1(X')\right\rangle+\left\langle\hat{S}_1^B(X')\right\rangle}$$

$$\frac{\delta}{\delta S^T(X,,\theta-2)}c$$
$$\simeq\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3$$
$$\times\left\{h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)+Z\right)+Z^2S(X)\right\}$$
$$+c\frac{\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\frac{\partial f(X)}{\partial S^T(X,\theta-2)}}+S_1^B(X,X)\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\left\{h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)+Z\right\}\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}$$

with:

$$Z=\frac{\frac{\hat{h}(X)}{2}}{1+\left(\hat{h}(X)\left(\frac{f(X)+\bar{r}(X)}{2}-\frac{\left\langle\hat{f}(X')\right\rangle_{\hat{h}_1}+\left\langle\hat{r}(X')\right\rangle_{\hat{h}_2}}{2}\right)\right)}$$

and:

$$c\frac{\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\frac{\partial f(X)}{\partial S^T(X,\theta-2)}}$$
$$\simeq S_1^B(X,X)\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\left\{h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)+Z\right\}\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}$$

$$d=S_E(X,X,\theta-1)\left(S_E^B(X,\theta-1)+S_L^B(X,\theta-1)\right)$$

$$\frac{\delta}{\delta\bar{f}(X,\theta-1)}d\simeq 2S_E(X,X,\theta-1)\,h_1^B(X)$$

$$e_0=\left(\frac{\left(1-\left(\hat{S}(X)\right)\right)\hat{S}_E(X)}{\left(1-\left(\hat{S}_E(X')\right)\right)^2}-\frac{\hat{S}(X)}{2\left(1-\left(\hat{S}_E(X')\right)\right)}\right)\left(\hat{f}(X)-\bar{r}\right)+S_E(X,X,\theta-1)$$

$$\frac{\delta}{\delta\bar{f}(X,\theta-1)}e\simeq 0$$

$$\frac{\delta}{\delta\bar{f}(X,\theta-1)}e_0\simeq 0$$

$$\frac{\delta}{\delta\hat{f}(X,\theta-1)}d\simeq 0$$

$$\frac{\delta}{\delta\hat{f}(X,\theta-1)}e\simeq\frac{\hat{S}(X)}{2\left(1-\left(\hat{S}_E(X')\right)\right)}\frac{1}{1+\frac{\Delta\hat{f}(X')+\Delta\hat{r}(X')}{2}}-\frac{\left(1-\left(\hat{S}(X)\right)\right)\hat{S}_E(X)}{\left(1-\left(\hat{S}_E(X')\right)\right)^2}\left(\frac{1}{1+\Delta\hat{f}(X)}\right)$$

$$\frac{\delta}{\delta\hat{f}\left(X,\theta-1\right)}e_0\simeq\frac{\left(1-\left(\hat{S}\left(X\right)\right)\right)\hat{S}_E\left(X\right)}{\left(1-\left(\hat{S}_E\left(X'\right)\right)\right)^2}-\frac{\hat{S}\left(X\right)}{2\left(1-\left(\hat{S}_E\left(X'\right)\right)\right)}$$

$$\begin{aligned}
&\frac{\delta}{\delta S^T\left(X,,\theta-2\right)}d\\
\simeq\quad&\left(\frac{\frac{\hat{h}(X)}{2}}{1+\left(\hat{h}\left(X\right)\left(\frac{f(X)+\bar{r}(X)}{2}-\frac{\left\langle\hat{f}(X')\right\rangle_{\hat{h}_1}+\left\langle\hat{r}(X')\right\rangle_{\hat{h}_2}}{2}\right)\right)}\right)\frac{\partial f\left(X\right)}{\partial S^T\left(X,\theta-2\right)}d\\
&+\left(h_1^B\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_1^B\left(X\right)\right\rangle\right)\right)\left(\frac{\partial f\left(X\right)}{\partial S^T\left(X,\theta-2\right)}\right)^2+\frac{\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\frac{\partial f(X)}{\partial S^T(X,\theta-2)}}d\\
\to\quad&\left(\frac{\frac{\hat{h}(X)}{2}d}{1+\left(\hat{h}\left(X\right)\left(\frac{f(X)+\bar{r}(X)}{2}-\frac{\left\langle\hat{f}(X')\right\rangle_{\hat{h}_1}+\left\langle\hat{r}(X')\right\rangle_{\hat{h}_2}}{2}\right)\right)}+\frac{\frac{\partial f(X)}{\partial S^T(X,\theta-2)}h_1^B\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_1^B\left(X\right)\right\rangle\right)}{S_1^B\left(X,\theta-1\right)+S_2^B\left(X,\theta-1\right)}\right)\\
&\times\frac{\partial f\left(X\right)}{\partial S^T\left(X,\theta-2\right)}+\frac{\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\frac{\partial f(X)}{\partial S^T(X,\theta-2)}}d
\end{aligned}$$

$$\frac{\delta}{\delta S^T\left(X,,\theta-2\right)}e\simeq 0$$

$$\begin{aligned}
\frac{\delta}{\delta S^T\left(X,,\theta-2\right)}e_0&\simeq\frac{\delta}{\delta S^T\left(X,,\theta-2\right)}S_E\left(X,X,\theta-1\right)\\
&\simeq\frac{\delta}{\delta f\left(X\right)}S_E\left(X,X,\theta-1\right)\frac{\partial f\left(X\right)}{\partial S^T\left(X,\theta-2\right)}\\
&\simeq h_E\left(X\right)\frac{\partial f\left(X\right)}{\partial S^T\left(X,\theta-2\right)}
\end{aligned}$$

$$\begin{aligned}
\frac{\delta}{\delta\bar{f}\left(X,\theta-1\right)}f&\simeq -S_1\left(X,X,\theta-1\right)\left(S_1^B\left(X,\theta-1\right)+S_2^B\left(X,\theta-1\right)\right)\\
&\quad\times\left[\frac{\partial f\left(X\right)}{\partial S^T\left(X,\theta-1\right)}\right]^2\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}\left(X\right)\right|^2}{\bar{K}_X\left|\bar{\Psi}\left(X\right)\right|^2}\frac{\partial_{f(X)}K_X\left|\Psi\left(X\right)\right|^2}{K_X\left|\Psi\left(X\right)\right|^2}\\
&\simeq 0
\end{aligned}$$

$$\begin{aligned}
&\frac{\delta}{\delta\hat{f}\left(X,\theta-1\right)}f\\
\simeq\quad&-\frac{\left(1-\left\langle\hat{S}\left(X'\right)\right\rangle\right)}{1-\left\langle\hat{S}_1\left(X'\right)\right\rangle}\frac{\left\langle\hat{h}\left(X',X\right)\right\rangle\left\langle h\left(X\right)\right\rangle}{4}\frac{\partial f\left(X\right)}{\partial S^T\left(X,\theta-1\right)}
\end{aligned}$$

$$\frac{\delta}{\delta\bar{f}\left(X,\theta-1\right)}a_0=\left(\left(2S_1^B\left(X,\theta-1\right)+S_2^B\left(X,\theta-1\right)\right)h_1^B\left(X\right)\left(\left\langle\bar{h}\left(X\right)\right\rangle+\left\langle\hat{h}_1^B\left(X\right)\right\rangle\right)\right)\frac{\partial f\left(X\right)}{\partial S^T\left(X,\theta-2\right)}$$

$$\frac{\delta}{\delta \bar{f}(X,\theta-1)} c = \left\{ h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle + \left\langle \hat{h}_1^B(X)\right\rangle\right) \frac{\partial f(X)}{\partial S^T(X,\theta-2)} - \frac{\left\langle \bar{h}(X',X)\right\rangle}{2} h_1^B(X) \frac{1-\left\langle \bar{S}(X')\right\rangle}{1-\left\langle \bar{S}_1(X')\right\rangle} \right\} \times \frac{\partial f(X)}{\partial S^T(X,\theta-2)}$$

$$\frac{\delta}{\delta S^T(X,,\theta-2)} c$$

$$\simeq \left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3 \times \left\{ h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle + \left\langle \hat{h}_1^B(X)\right\rangle\right)\left(h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle + \left\langle \hat{h}_1^B(X)\right\rangle\right) + Z\right) + Z^2 S(X)\right\}$$
$$+ c\frac{\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\frac{\partial f(X)}{\partial S^T(X,\theta-2)}} + S_1^B(X,X)\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\left\{ h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle + \left\langle \hat{h}_1^B(X)\right\rangle\right) + Z\right\}\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}$$

$$\frac{\delta}{\delta S^T(X,,\theta-2)} d$$

$$\simeq \left(\frac{\frac{\hat{h}(X)}{2}}{1+\left(\hat{h}(X)\left(\frac{f(X)+\bar{r}(X)}{2} - \frac{\left\langle \hat{f}(X')\right\rangle_{\hat{h}_1} + \left\langle \hat{r}(X')\right\rangle_{\hat{h}_2}}{2}\right)\right)}\right)\frac{\partial f(X)}{\partial S^T(X,\theta-2)} d$$
$$+ \left(h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle + \left\langle \hat{h}_1^B(X)\right\rangle\right)\right)\left(\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\right)^2 + \frac{\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\frac{\partial f(X)}{\partial S^T(X,\theta-2)}} d$$
$$\rightarrow \left(\frac{\frac{\hat{h}(X)}{2} d}{1+\left(\hat{h}(X)\left(\frac{f(X)+\bar{r}(X)}{2} - \frac{\left\langle \hat{f}(X')\right\rangle_{\hat{h}_1} + \left\langle \hat{r}(X')\right\rangle_{\hat{h}_2}}{2}\right)\right)} + \frac{\frac{\partial f(X)}{\partial S^T(X,\theta-2)} h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle + \left\langle \hat{h}_1^B(X)\right\rangle\right)}{S_1^B(X,\theta-1) + S_2^B(X,\theta-1)}\right)$$
$$\times \frac{\partial f(X)}{\partial S^T(X,\theta-2)} + \frac{\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\frac{\partial f(X)}{\partial S^T(X,\theta-2)}} d$$

$$\frac{\delta}{\delta S^T(X,,\theta-2)} e \simeq 0$$

$$\begin{aligned}
\frac{\delta}{\delta S^T(X,,\theta-2)} e_0 &\simeq \frac{\delta}{\delta S^T(X,,\theta-2)} S_E(X,X,\theta-1) \\
&\simeq \frac{\delta}{\delta f(X)} S_E(X,X,\theta-1)\frac{\partial f(X)}{\partial S^T(X,\theta-2)} \\
&\simeq h_E(X)\frac{\partial f(X)}{\partial S^T(X,\theta-2)}
\end{aligned}$$

$$\frac{\delta}{\delta S^T(X,,\theta-2)} f$$

$$\simeq S_1(X,X,\theta-1)\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3 \frac{\frac{\hat{h}(X)}{2}}{1+\left(\hat{h}(X)\left(\frac{f(X)+\bar{r}(X)}{2}-\frac{\left\langle \hat{f}(X')\right\rangle_{\hat{h}_1}+\langle \hat{r}(X')\rangle_{\hat{h}_2}}{2}\right)\right)}$$

$$\times\left(h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle+\left\langle \hat{h}_1^B(X)\right\rangle\right)+ZS(X)\right)$$

$$+2S_1(X,X,\theta-1)\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}\left(h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle+\left\langle \hat{h}_1^B(X)\right\rangle\right)+Z\right)$$

$$\frac{\delta}{\delta \hat{f}(X,\theta-1)} d \simeq 0$$

$$\frac{\delta}{\delta \bar{f}(X,\theta-1)} h \simeq 0$$

$$\frac{\delta}{\delta \hat{f}(X,\theta-1)} h \simeq 0$$

$$\frac{\delta}{\delta S^T(X,,\theta-2)} h \simeq \frac{\partial f(X,\theta-1)}{\partial S^T(X,,\theta-2)} h(X)$$

$$\frac{\delta}{\delta \bar{f}(X,\theta-1)} g \simeq 0$$

$$\frac{\delta}{\delta \hat{f}(X,\theta-1)} g \simeq 0$$

$$\frac{\delta}{\delta S^T(X,,\theta-2)} g \simeq 0$$

$$\frac{\delta}{\delta \bar{f}(X,\theta-1)} i \simeq 0$$

$$\frac{\delta}{\delta \hat{f}(X,\theta-1)} i \simeq 0$$

$$\frac{\delta}{\delta S^T(X,,\theta-2)} i \simeq S(X)\left(Z^2 S(X)\right)\left[\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\right]^2 + i\frac{\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\frac{\partial f(X)}{\partial S^T(X,\theta-2)}}$$

Given that the derivatives of coefficients with respect to $\bar{K}_X\left|\bar{\Psi}(X)\right|^2$, $\hat{K}_X\left|\hat{\Psi}(X)\right|^2$ and $K_X\left|\Psi(X)\right|^2$ are approximatively the coefficients divided by $\bar{K}_X\left|\bar{\Psi}(X)\right|^2$, $\hat{K}_X\left|\hat{\Psi}(X)\right|^2$ and $K_X\left|\Psi(X)\right|^2$ we can neglect the derivatives

$$\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2} \simeq -\frac{2}{\left(1-\left\langle \bar{S}\right\rangle\right)\bar{f}(X)+\left\langle \bar{S}(X',X)\right\rangle_{X'}\left\langle \bar{f}\right\rangle}$$

$$\frac{\partial_{\hat{f}(X)}\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{\hat{K}_X\left|\hat{\Psi}(X)\right|^2} \simeq -\frac{2}{\hat{f}(X)+\frac{\left(\left\langle\hat{S}_E^B(X,X')\right\rangle_{X'}+\left\langle\hat{S}_L^B(X,X')\right\rangle_{X'}\right)\frac{\langle\bar{K}\rangle\|\bar{\Psi}\|^2}{\langle\hat{K}\rangle\|\hat{\Psi}\|^2}\langle\bar{f}\rangle}{1-\langle\bar{S}\rangle}}$$

$$\frac{\partial_{\bar{f}(X)}\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{\hat{K}_X\left|\hat{\Psi}(X)\right|^2} \simeq -\frac{2\frac{\left(\left\langle\hat{S}_E^B(X,X')\right\rangle_{X'}+\left\langle\hat{S}_L^B(X,X')\right\rangle_{X'}\right)\frac{\langle\bar{K}\rangle\|\bar{\Psi}\|^2}{\langle\hat{K}\rangle\|\hat{\Psi}\|^2}\langle\bar{f}\rangle}{1-\langle\bar{S}\rangle}}{\hat{f}(X)+\frac{\left(\left\langle\hat{S}_E^B(X,X')\right\rangle_{X'}+\left\langle\hat{S}_L^B(X,X')\right\rangle_{X'}\right)\frac{\langle\bar{K}\rangle\|\bar{\Psi}\|^2}{\langle\hat{K}\rangle\|\hat{\Psi}\|^2}\langle\bar{f}\rangle}{1-\langle\bar{S}\rangle}} << 1$$

$$\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2} \simeq \frac{2}{r}\frac{1}{f(X)+C_0}$$

As a consequence, the block involving disposable capital can be wrtten as:

$$\begin{pmatrix} \frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2} & 0 & 0 \\ \frac{\partial_{\bar{f}(X)}\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{\hat{K}_X\left|\hat{\Psi}(X)\right|^2} & \frac{\partial_{\hat{f}(X)}\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{\hat{K}_X\left|\hat{\Psi}(X)\right|^2} & 0 \\ 0 & 0 & \frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\frac{\partial f(X)}{\partial S^T(X,\theta-2)} \end{pmatrix}$$

$$\simeq \begin{pmatrix} -\frac{2}{\left(1-\langle\bar{S}\rangle\right)\bar{f}(X)+\left\langle\bar{S}(X',X)\right\rangle_{X'}\langle\bar{f}\rangle} & 0 & 0 \\ 0 & -\frac{2}{\hat{f}(X)+\frac{\left(\left\langle\hat{S}_E^B(X,X')\right\rangle_{X'}+\left\langle\hat{S}_L^B(X,X')\right\rangle_{X'}\right)\frac{\langle\bar{K}\rangle\|\bar{\Psi}\|^2}{\langle\hat{K}\rangle\|\hat{\Psi}\|^2}\langle\bar{f}\rangle}{1-\langle\bar{S}\rangle}} & 0 \\ 0 & 0 & -\frac{2}{r}\frac{1}{f(X)+C_0}\frac{\partial f(X)}{\partial S^T(X,\theta-2)} \end{pmatrix}$$

A5.1.3.2 Estimation of the quadratic matrices The matrices involved in the quadratic term are:

$$\bar{M} = \begin{bmatrix} \frac{\delta}{\delta f(X,\theta-1)}a & 0 & \frac{\bar{h}(X',X)}{2}\frac{\delta}{\delta f(X,\theta-1)}c & \frac{\delta(a_0+a_1)}{\delta f(X,\theta-1)} & 0 & -\frac{\delta a_0}{\delta f(X,\theta-1)} \\ 0 & 0 & \frac{\delta}{\delta\hat{f}(X,\theta-1)}c & 0 & 0 & 0 \\ 0 & 0 & \frac{\delta}{\delta S^T(X,,\theta-2)}c & 0 & 0 & 0 \\ & & 0\,[3\times 6] & & & \end{bmatrix}$$

$$,\hat{M} = \begin{pmatrix} 0 & 0 & 0 & \frac{\delta}{\delta f(X,\theta-1)}d & 0 & -\frac{\delta}{\delta f(X,\theta-1)}d \\ 0 & 0 & \frac{\delta}{\delta\hat{f}(X,\theta-1)}f & 0 & \frac{\delta}{\delta\hat{f}(X,\theta-1)}e_0 & 0 \\ 0 & 0 & \frac{\delta}{\delta S^T(X,,\theta-2)}f & \frac{\delta}{\delta S^T(X,,\theta-2)}d & \frac{\delta}{\delta S^T(X,,\theta-2)}e_0 & -\frac{\delta}{\delta S^T(X,,\theta-2)}d \\ & & & 0\,[3\times 6] & & \end{pmatrix}$$

$$M_{3,3}^T = \frac{\delta}{\delta S^T(X,,\theta-2)}i,\ M_{3,5}^T = \frac{\partial}{\partial S^T(X,,\theta-2)}h\ ,M_{i,j}^T = 0 \text{ otherwise}$$

$$M_{\bar{K},1,1} = \partial_{\bar{f}(X)}\left(\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}\right),\ M_{\bar{K},i,j} = 0 \text{ otherwise}$$

$$M_{\hat{K},2,2} = \partial_{\hat{f}(X)}\left(\frac{\partial_{\hat{f}(X)}\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}\right),\ M_{\hat{K},i,j} = 0 \text{ otherwise}$$

with:

$$\partial_{\bar{f}(X)}\left(\frac{\partial_{\bar{f}(X)}\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}\right)\simeq\frac{6}{\left(\bar{f}(X)\right)^4}$$

$$\partial_{\hat{f}(X)}\left(\frac{\partial_{\hat{f}(X)}\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}\right)\simeq\frac{6}{\left(\hat{f}(X)\right)^4}$$

$$\begin{aligned}M_{K,3,3} &= \partial_{f(X)}\left(\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\right)\left(\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\right)^2\\ &\quad+\frac{\partial_{f(X)}K_X\left|\Psi(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}\\ &\simeq -\frac{2}{r}\frac{1}{\left(f(X)+C_0\right)^2}\left(\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\right)^3\end{aligned}$$

and $M_{K,i,j}=0$ otherwise.

13.1.2 A5.1.4 Stability condition

Fo relatively large fluctuations, the stability condition becomes:

$$\frac{\left(\delta\mathbf{X}^t\bar{M}\delta\mathbf{X}\right)^2+\left(\delta\mathbf{X}^t\hat{M}\delta\mathbf{X}\right)^2+\left(\delta\mathbf{X}^tM^T\delta\mathbf{X}\right)^2+\left(\delta\mathbf{X}^tM_{\bar{K}}\delta\mathbf{X}\right)^2+\left(\delta\mathbf{X}^tM_{\hat{K}}\delta\mathbf{X}\right)^2+\left(\delta\mathbf{X}^tM_K\delta\mathbf{X}\right)^2}{\left(\delta\bar{S}(X)\right)^2+\left(\delta\hat{S}(X)\right)^2+\left(\delta S^T(X)\right)^2+\left(\delta\bar{K}_X\left|\bar{\Psi}(X)\right|^2\right)^2+\left(\delta\hat{K}_X\left|\hat{\Psi}(X)\right|^2\right)^2+\left(\delta K_X\left|\Psi(X)\right|^2\right)^2}<1-\left\langle\lambda^2\right\rangle \tag{203}$$

$$
\begin{aligned}
&\left(1-\left\langle \lambda^{2}\right\rangle\right)\left(\left(\delta\bar{S}(X)\right)^{2}+\left(\delta\hat{S}(X)\right)^{2}+\left(\delta S^{T}(X)\right)^{2}+\left(\delta\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}\right)^{2}+\left(\delta\hat{K}_{X}\left|\hat{\Psi}(X)\right|^{2}\right)^{2}+\left(\delta K_{X}\left|\Psi(X)\right|^{2}\right)^{2}\right)\\
> &\left\{\frac{\left(\delta\bar{S}(X)\right)^{2}\delta a}{\delta\bar{f}(X,\theta-1)}+\delta\bar{S}(X)\left(\frac{\bar{h}(X',X)}{2}\frac{\delta S^{T}(X)\delta c}{\delta\bar{f}(X,\theta-1)}+\frac{\delta\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}\delta a_{1}}{\delta\bar{f}(X,\theta-1)}+\frac{\delta a_{0}}{\delta\bar{f}(X,\theta-1)}\frac{\delta\frac{\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}}{\frac{\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}}\right)\right.\\
&\left.+\left(\delta\bar{S}(X)\frac{\delta}{\delta\hat{f}(X,\theta-1)}c+\delta\hat{S}(X)\frac{\delta}{\delta S^{T}(X,,\theta-2)}c\right)\delta S^{T}(X)\right\}^{2}\\
&+\left\{\delta\bar{S}(X)\frac{\delta d}{\delta\bar{f}(X,\theta-1)}\frac{\delta\frac{\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}}{\frac{\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}}+\delta\hat{S}(X)\left(\frac{\delta f}{\delta\hat{f}(X,\theta-1)}\delta\bar{S}(X)+\frac{\delta e_{0}}{\delta\hat{f}(X,\theta-1)}\delta\hat{K}_{X}\left|\hat{\Psi}(X)\right|^{2}\right)\right.\\
&\left.+\delta S^{T}(X)\left(\frac{\delta f}{\delta S^{T}(X,,\theta-2)}\delta S^{T}(X)+\frac{\delta e_{0}}{\delta S^{T}(X,,\theta-2)}\delta\hat{K}_{X}\left|\hat{\Psi}(X)\right|^{2}+\frac{\delta d}{\delta S^{T}(X,,\theta-2)}\frac{\delta\frac{\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}}{\frac{\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}}\right)\right\}^{2}\\
&+\left\{\delta S^{T}(X)\left(\frac{\delta d}{\delta S^{T}(X,,\theta-2)}\frac{\delta\frac{\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}}{\frac{\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}}+\frac{\delta f}{\delta S^{T}(X,,\theta-2)}\delta S^{T}(X)+\frac{\delta e_{0}}{\delta S^{T}(X,,\theta-2)}\delta\hat{K}_{X}\left|\hat{\Psi}(X)\right|^{2}\right)\right\}^{2}\\
&+\left\{\frac{\delta}{\delta S^{T}(X,,\theta-2)}i\left(\delta S^{T}(X)\right)^{2}+\frac{\partial h}{\partial S^{T}(X,,\theta-2)}\delta S^{T}(X)\delta\hat{K}_{X}\left|\hat{\Psi}(X)\right|^{2}\right\}^{2}\\
&+\left\{\partial_{\bar{f}(X)}\left(\frac{\partial_{\bar{f}(X)}\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}{\bar{K}_{X}\left|\bar{\Psi}(X)\right|^{2}}\right)\left(\delta\bar{S}(X)\right)^{2}\right\}^{2}+\left\{\partial_{\hat{f}(X)}\left(\frac{\partial_{\hat{f}(X)}\hat{K}_{X}\left|\hat{\Psi}(X)\right|^{2}}{\hat{K}_{X}\left|\hat{\Psi}(X)\right|^{2}}\right)\left(\delta\hat{S}(X)\right)^{2}\right\}^{2}\\
&+\left\{\left(\partial_{f(X)}\left(\frac{\partial_{f(X)}K_{X}\left|\Psi(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}\right)\left(\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}\right)^{2}+\frac{\partial_{f(X)}K_{X}\left|\Psi(X)\right|^{2}}{K_{X}\left|\Psi(X)\right|^{2}}\frac{\partial^{2}f(X)}{\partial\left(S^{T}(X,\theta-2)\right)^{2}}\right)\left(\delta S^{T}(X)\right)^{2}\right\}^{2}
\end{aligned}
$$

We assume as in Appendix 4 that $\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}>>1$.

The coefficients to consider in first approximation are:

$$
\frac{\delta}{\delta\bar{f}(X,\theta-1)}a_{0}=\left(\left(2S_{1}^{B}(X,\theta-1)+S_{2}^{B}(X,\theta-1)\right)h_{1}^{B}(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_{1}^{B}(X)\right\rangle\right)\right)\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}
$$

$$
\frac{\delta}{\delta\bar{f}(X,\theta-1)}c\simeq h_{1}^{B}(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_{1}^{B}(X)\right\rangle\right)\left(\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}\right)^{2}
$$

$$
\frac{\delta}{\delta S^{T}(X,,\theta-2)}e_{0}\simeq h_{E}(X)\frac{\partial f(X)}{\partial S^{T}(X,\theta-2)}
$$

$$
\begin{aligned}
&\frac{\delta}{\delta S^{T}(X,,\theta-2)}c\\
\simeq &\left(\frac{\partial f(X)}{\partial S^{T}(X,\theta-1)}\right)^{3}\\
&\times\left\{h_{1}^{B}(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_{1}^{B}(X)\right\rangle\right)\left(h_{1}^{B}(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_{1}^{B}(X)\right\rangle\right)+Z\right)+Z^{2}S(X)\right\}
\end{aligned}
$$

$$\frac{\delta}{\delta S^{T}(X,,\theta-2)}d$$
$$\simeq \frac{h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle+\left\langle \hat{h}_1^B(X)\right\rangle\right)}{S_1^B(X,\theta-1)+S_2^B(X,\theta-1)}\left(\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\right)^2$$

$$\frac{\delta}{\delta S^{T}(X,,\theta-2)}f$$
$$\simeq S_1(X,X,\theta-1)\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3\frac{\frac{\hat{h}(X)}{2}\left(h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle+\left\langle \hat{h}_1^B(X)\right\rangle\right)+ZS(X)\right)}{1+\left(\hat{h}(X)\left(\frac{f(X)+\bar{r}(X)}{2}-\frac{\left\langle \hat{f}(X')\right\rangle_{\hat{h}_1}+\left\langle \hat{r}(X')\right\rangle_{\hat{h}_2}}{2}\right)\right)}$$

$$\frac{\delta}{\delta S^{T}(X,,\theta-2)}h\simeq\frac{\partial f(X,\theta-1)}{\partial S^T(X,,\theta-2)}h(X)$$

$$\frac{\delta}{\delta S^{T}(X,,\theta-2)}i\simeq S(X)\left(Z^2S(X)\right)\left[\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\right]^2+i\frac{\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}}{\frac{\partial f(X)}{\partial S^T(X,\theta-2)}}$$

The dominant contributions involving the fluctuations in $\bar{K}_X\left|\bar{\Psi}(X)\right|^2$, $\hat{K}_X\left|\hat{\Psi}(X)\right|^2$ and $K_X\left|\Psi(X)\right|^2$ are:

$$\left\{\frac{\left(\delta\bar{S}(X)\right)^2\delta a}{\delta\bar{f}(X,\theta-1)}+\delta\bar{S}(X)\left(\frac{\bar{h}(X',X)}{2}\frac{\delta S^T(X)\delta c}{\delta\bar{f}(X,\theta-1)}+\frac{\delta\bar{K}_X\left|\bar{\Psi}(X)\right|^2\delta a_1}{\delta\bar{f}(X,\theta-1)}+\frac{\delta a_0}{\delta\bar{f}(X,\theta-1)}\frac{\delta\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}}{\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}}\right)\right.$$
$$\left.+\left(\delta\bar{S}(X)\frac{\delta}{\delta\hat{f}(X,\theta-1)}c+\delta\hat{S}(X)\frac{\delta}{\delta S^T(X,,\theta-2)}c\right)\delta S^T(X)\right\}^2$$
$$\to\left\{\delta\bar{S}(X)h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle+\left\langle \hat{h}_1^B(X)\right\rangle\right)\left(\left\{\left(h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle+\left\langle \hat{h}_1^B(X)\right\rangle\right)+Z\right)+Z^2S(X)\right\}\right.\right.$$
$$\left.\left.\times\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3\delta S^T(X)+\left(2S_1^B(X,\theta-1)+S_2^B(X,\theta-1)\right)\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\frac{\delta\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}}{\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}}\right)\right\}^2$$
$$\to\left(h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle+\left\langle \hat{h}_1^B(X)\right\rangle\right)\left(\left(h_1^B(X)\left(\left\langle \bar{h}(X)\right\rangle+\left\langle \hat{h}_1^B(X)\right\rangle\right)+Z\right)+Z^2S(X)\right)\right)^2$$
$$\times\left(\delta\bar{S}(X)\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3\delta S^T(X)\right)^2$$

$$\left\{\delta\bar{S}(X)\frac{\delta d}{\delta\bar{f}(X,\theta-1)}\frac{\delta\frac{\bar{K}_X|\bar{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}{\frac{\bar{K}_X|\bar{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}+\delta\hat{S}(X)\left(\frac{\delta f}{\delta\hat{f}(X,\theta-1)}\delta\bar{S}(X)+\frac{\delta e_0}{\delta\hat{f}(X,\theta-1)}\delta\hat{K}_X\left|\hat{\Psi}(X)\right|^2\right)\right.$$

$$\left.+\delta S^T(X)\left(\frac{\delta f}{\delta S^T(X,,\theta-2)}\delta S^T(X)+\frac{\delta e_0}{\delta S^T(X,,\theta-2)}\delta\hat{K}_X\left|\hat{\Psi}(X)\right|^2+\frac{\delta d}{\delta S^T(X,,\theta-2)}\frac{\delta\frac{\bar{K}_X|\bar{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}{\frac{\bar{K}_X|\bar{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}\right)\right\}^2$$

$$\rightarrow\left\{\delta S^T(X)\left(S_1(X,X,\theta-1)\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3\frac{\frac{\hat{h}(X)}{2}\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)+ZS(X)\right)}{1+\left(\hat{h}(X)\left(\frac{f(X)+\bar{r}(X)}{2}-\frac{\left\langle\hat{f}(X')\right\rangle_{\hat{h}_1}+\left\langle\hat{r}(X')\right\rangle_{\hat{h}_2}}{2}\right)\right)}\delta S^T(X)\right.\right.$$

$$\left.\left.+\frac{h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)}{S_1^B(X,\theta-1)+S_2^B(X,\theta-1)}\left(\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\right)^2\frac{\delta\frac{\bar{K}_X|\bar{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}{\frac{\bar{K}_X|\bar{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}\right)\right\}^2$$

$$\rightarrow\left(\delta S^T(X)\,S_1(X,X,\theta-1)\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3\frac{\frac{\hat{h}(X)}{2}\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)+ZS(X)\right)}{1+\left(\hat{h}(X)\left(\frac{f(X)+\bar{r}(X)}{2}-\frac{\left\langle\hat{f}(X')\right\rangle_{\hat{h}_1}+\left\langle\hat{r}(X')\right\rangle_{\hat{h}_2}}{2}\right)\right)}\delta S^T(X)\right)^2$$

$$\left\{\frac{\delta}{\delta S^T(X,,\theta-2)}i\left(\delta S^T(X)\right)^2+\frac{\partial h}{\partial S^T(X,,\theta-2)}\delta S^T(X)\,\delta\hat{K}_X\left|\hat{\Psi}(X)\right|^2\right\}^2$$

$$\rightarrow\left\{S(X)\left(Z^2S(X)\right)\left[\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\right]^2\left(\delta S^T(X)\right)^2\right\}^2$$

The four last terms can be replaced by:

$$\left(\delta S^T(X)\frac{\frac{2}{r}}{\left(f(X)+C_0\right)^2}\left(\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\right)^3\delta S^T(X)\right)^2$$

and we find the condition:

$$\left(1-\left\langle\lambda^2\right\rangle\right)\left(\left(\delta\bar{S}(X)\right)^2+\left(\delta\hat{S}(X)\right)^2+\left(\delta S^T(X)\right)^2+\left(\delta\bar{K}_X\left|\bar{\Psi}(X)\right|^2\right)^2+\left(\delta\hat{K}_X\left|\hat{\Psi}(X)\right|^2\right)^2+\left(\delta K_X\left|\Psi(X)\right|^2\right)^2\right)$$

$$>\ A\left(\delta\bar{S}(X)\,\delta S^T(X)\right)^2+\left(B_1\left(\delta S^T(X)\right)^2+B_2\delta S^T(X)\frac{\delta\frac{\bar{K}_X|\bar{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}{\frac{\bar{K}_X|\bar{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}\right)^2+C\left(\left(\delta S^T(X)\right)^2\right)^2$$

wth:

$$A=\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)\left(\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)+Z\right)+Z^2S(X)\right)\right)^2\times\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^6$$

$$B_1=S_1(X,X,\theta-1)\left(\frac{\partial f(X)}{\partial S^T(X,\theta-1)}\right)^3\frac{\frac{\hat{h}(X)}{2}\left(h_1^B(X)\left(\left\langle\bar{h}(X)\right\rangle+\left\langle\hat{h}_1^B(X)\right\rangle\right)+ZS(X)\right)}{1+\left(\hat{h}(X)\left(\frac{f(X)+\bar{r}(X)}{2}-\frac{\left\langle\hat{f}(X')\right\rangle_{\hat{h}_1}+\left\langle\hat{r}(X')\right\rangle_{\hat{h}_2}}{2}\right)\right)}$$

$$B_2 = \frac{h_1^B(X)\left(\langle \bar{h}(X)\rangle + \left\langle \hat{h}_1^B(X)\right\rangle\right)}{S_1^B(X,\theta-1)+S_2^B(X,\theta-1)}\left(\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\right)^2 \frac{\delta\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}{\frac{\bar{K}_X\left|\bar{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}$$

$$C = \left(\left(Z^2 S(X)\right)\left[\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\right]^2\right)^2 + \left(\delta S^T(X)\frac{\frac{2}{r}}{(f(X)+C_0)^2}\left(\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\right)^3 \delta S^T(X)\right)^2$$

A5.2 Investors

The second order equation is:

$$\begin{pmatrix} \delta\hat{S}(X',X,\theta) \\ \delta S(X,,\theta-1) \end{pmatrix} = \begin{bmatrix} \alpha & \frac{\hat{h}(X',X)}{2}c \\ \frac{h}{\frac{\hat{h}(X',X)}{2}} & \beta \end{bmatrix}\begin{pmatrix} \delta\hat{S}(X',X,\theta-1) \\ \delta S(X,,\theta-2) \end{pmatrix} + \begin{pmatrix} \delta\hat{S}(X',X,\theta) \\ \delta S(X,,\theta-1) \end{pmatrix}^t \begin{bmatrix} \hat{M} \\ M \end{bmatrix}\begin{pmatrix} \delta\hat{S}(X',X,\theta) \\ \delta S(X,\theta-1) \end{pmatrix}$$

replaced by:

$$\begin{pmatrix} \delta\hat{S}(X',X,\theta) \\ \delta S(X,,\theta-1) \\ \frac{\delta\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{\hat{K}_X\left|\hat{\Psi}(X)\right|^2} \\ \frac{\delta K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2} \end{pmatrix}$$

$$= \begin{bmatrix} \alpha & \frac{\hat{h}(X',X)}{2}c & \alpha_0+c_0 & \alpha_1-c_0 \\ \frac{h}{\frac{\hat{h}(X',X)}{2}} & \beta & \beta_0+h & -\beta_0-h \\ \frac{\partial_{\hat{f}(X,\theta-1)}\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{\hat{K}_X\left|\hat{\Psi}(X)\right|^2} & 0 & 0 & 0 \\ 0 & \frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\frac{\partial f(X)}{\partial S(X,,\theta-2)} & 0 & 0 \end{bmatrix}\begin{pmatrix} \delta\hat{S}(X',X,\theta-1) \\ \delta S(X,,\theta-2) \\ \frac{\delta\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{\hat{K}_X\left|\hat{\Psi}(X)\right|^2} \\ \frac{\delta K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2} \end{pmatrix}$$

$$+\begin{pmatrix} \delta\hat{S}(X',X,\theta) \\ \delta S(X,,\theta-1) \\ \frac{\delta\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{\hat{K}_X\left|\hat{\Psi}(X)\right|^2} \\ \frac{\delta K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2} \end{pmatrix}^t \begin{bmatrix} \hat{M} \\ M \\ M_{\hat{K}} \\ M_K \end{bmatrix}\begin{pmatrix} \delta\hat{S}(X',X,\theta-1) \\ \delta S(X,,\theta-2) \\ \frac{\delta\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{\hat{K}_X\left|\hat{\Psi}(X)\right|^2} \\ \frac{\delta K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2} \end{pmatrix}$$

The coefficients and th matrices $\hat{M}$, M, $M_{\hat{K}}$, M_K are defined below.

A5.2.1 Coefficients

The coefficients are given by:

$$\alpha = \left(\frac{\hat{S}(X)}{\left(1-\left(\hat{S}(X)\right)\right)}\frac{1}{2\left(1+\frac{\Delta\hat{f}(X')+\Delta\hat{r}(X')}{2}\right)} - \frac{\hat{S}_1(X)}{\left(1-\left(\hat{S}_1(X')\right)\right)}\frac{1}{1+\Delta\hat{f}(X)}\right)\left(\hat{f}(X)-\bar{r}\right) \quad (204)$$

$$\alpha_0 = \left(\frac{\hat{S}_1(X)}{\left(1-\left(\hat{S}_1(X')\right)\right)} - \frac{\hat{S}(X)}{\left(1-\left(\hat{S}(X)\right)\right)}\right)\left(\hat{f}(X)-\bar{r}\right) + \frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)} S_1(X,X,\theta-1) S(X) \tag{205}$$

$$\alpha_1 = -\frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)} S_1(X,X,\theta-1) S(X)$$

$$\beta = \left(\frac{\frac{\hat{h}(X)}{2} S(X)}{1+\left(\hat{h}(X)\left(\frac{f(X)+\bar{r}(X)}{2} - \frac{\left\langle \hat{f}(X')\right\rangle_{\hat{h}_1} + \langle \hat{r}(X')\rangle_{\hat{h}_2}}{2}\right)\right)}\right)\frac{\partial f(X)}{\partial S(X,\theta-2)} \simeq \frac{\hat{h}(X)}{2} S(X)\frac{\partial f(X)}{\partial S(X,\theta-2)}$$

$$\beta_0 \simeq S(X)$$

$$c = \left\{ -\frac{\left(1-\left\langle \hat{S}(X')\right\rangle\right)\left(\left\langle \hat{f}(X')\right\rangle - \bar{r}\right)}{1-\left\langle \hat{S}_1(X')\right\rangle}\frac{\left\langle \hat{h}(X',X)\right\rangle \langle h(X)\rangle}{4} + \frac{\frac{h(X)}{4}\left(1+\hat{h}(X)\right) S_1(X)\left(f(X)-\bar{r}\right)}{1+\left(\hat{h}(X)\left(f(X) - \frac{\left\langle \hat{f}(X')\right\rangle_{\hat{h}_1} + \langle \hat{r}(X')\rangle_{\hat{h}_2}}{2}\right) + \frac{h(X)}{2}\left(f(X)-\bar{r}(X)\right)\right)} + S_1(X)\left(\frac{\frac{\hat{h}(X)}{2} S(X)}{1+\left(\hat{h}(X)\left(\frac{f(X)+\bar{r}(X)}{2} - \frac{\left\langle \hat{f}(X')\right\rangle_{\hat{h}_1} + \langle \hat{r}(X')\rangle_{\hat{h}_2}}{2}\right)\right)}\right)\frac{\partial f(X)}{\partial S(X,\theta-2)}\right\} \times \frac{\partial f(X)}{\partial S(X,\theta-1)}$$

$$c_0 = S_1(X) S(X)\frac{\partial f(X)}{\partial S(X,\theta-1)}$$

$$h = S(X)$$

13.1.3 A5.2.2 Eigenvalues

The eigenvalues equation:

$$\det\left(\left(\begin{array}{cccc} 0 & cz^2 & vz & -vz \\ 0 & \beta & w & -w \\ s & 0 & 0 & 0 \\ 0 & tz & 0 & 0 \end{array}\right) - X\left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)\right) = 0$$

$$X^4 - X^3\beta - X^2 svz + X^2 twz + Xsvz\beta - Xcswz^2 = 0$$

that is:

$$X^3 - cswz^2 = 0$$

with a solution:

$$X < 0$$

A5.2.3 Estimation of coefficients derivatives and matrices

The variations of these coefficients are obtained by using the following formula derived in Gosselin and Lotz (2025a).

$$\begin{aligned}&\delta \hat{S}_1 \left(X', X, \theta - 1\right)\\ \rightarrow\quad &\frac{\hat{h}\left(X', X\right)}{2}\left(1 + \left(\delta \hat{f}\left(X'\right) - \frac{\left\langle h\left(X\right)\right\rangle}{2}\delta f\left(X\right)\right)\right)\end{aligned}$$

$$\delta \hat{S}_1 \left(X, \theta - 1\right) \simeq \left(\delta \hat{f}\left(X, \theta - 1\right) - \frac{\delta \hat{K}_{X'}\left|\hat{\Psi}\left(X'\right)\right|^2}{\hat{K}_{X'}\left|\hat{\Psi}\left(X'\right)\right|^2}\right)\hat{S}_1\left(X\right)$$

$$\delta \hat{S}\left(X, \theta - 1\right) \simeq \left(\frac{1}{2}\delta \hat{f}\left(X, \theta - 1\right) - \frac{\delta \hat{K}_{X'}\left|\hat{\Psi}\left(X'\right)\right|^2}{\hat{K}_{X'}\left|\hat{\Psi}\left(X'\right)\right|^2}\right)\hat{S}\left(X\right)$$

$$\begin{aligned}&\delta S_1\left(X, X, \theta - 1\right)\\ \rightarrow\quad &\frac{h\left(X\right)}{2}\left(\hat{h}\left(X\right) + \frac{h\left(X\right)}{2}\right)S_1\left(X, X\right)\delta f\left(X, \theta - 1\right) + \frac{\delta\left(\frac{\hat{K}_X\left|\hat{\Psi}\left(X\right)\right|^2}{K_X\left|\Psi\left(X\right)\right|^2}\right)S_1\left(X\right)}{\frac{\hat{K}_X\left|\hat{\Psi}\left(X\right)\right|^2}{K_X\left|\Psi\left(X\right)\right|^2}}\\ \rightarrow\quad &\frac{\delta\left(\frac{\hat{K}_X\left|\hat{\Psi}\left(X\right)\right|^2}{K_X\left|\Psi\left(X\right)\right|^2}\right)S_1\left(X\right)}{\frac{\hat{K}_X\left|\hat{\Psi}\left(X\right)\right|^2}{K_X\left|\Psi\left(X\right)\right|^2}} + \frac{h\left(X\right)}{2}\left(\hat{h}\left(X\right) + \frac{h\left(X\right)}{2}\right)S_1\left(X, X\right)\frac{\partial f\left(X\right)}{\partial S\left(X, \theta - 2\right)}\delta S\left(X, \theta - 2\right)\end{aligned}$$

$$\begin{aligned}\delta S\left(X, \theta - 1\right) \quad&\simeq\quad \frac{\hat{h}\left(X\right)}{2}\delta f\left(X, \theta - 1\right)S\left(X\right) + \frac{\delta\frac{\hat{K}_X\left|\hat{\Psi}\left(X\right)\right|^2}{K_X\left|\Psi\left(X\right)\right|^2}}{\frac{\hat{K}_X\left|\hat{\Psi}\left(X\right)\right|^2}{K_X\left|\Psi\left(X\right)\right|^2}}S\left(X\right)\\ &\simeq\quad \frac{\hat{h}\left(X\right)}{2}S\left(X\right)\frac{\partial f\left(X\right)}{\partial S\left(X, \theta - 2\right)}\delta S\left(X, \theta - 2\right) + \frac{\delta\frac{\hat{K}_X\left|\hat{\Psi}\left(X\right)\right|^2}{K_X\left|\Psi\left(X\right)\right|^2}}{\frac{\hat{K}_X\left|\hat{\Psi}\left(X\right)\right|^2}{K_X\left|\Psi\left(X\right)\right|^2}}S\left(X\right)\end{aligned}$$

We use in first approximation that:

$$\begin{aligned}\delta\frac{1 - \left(\hat{S}_1\left(X\right)\right)}{1 - \left(\hat{S}\left(X\right)\right)} \quad&\simeq\quad \delta\frac{1 - \frac{1}{2}\hat{S}\left(X\right)}{1 - \hat{S}\left(X\right)}\\ &\rightarrow\quad \frac{1}{2\left(1 - \hat{S}\left(X\right)\right)^2}\delta\hat{S}\left(X\right)\\ &\simeq\quad \frac{1}{2\left(1 - \hat{S}\left(X\right)\right)^2}\left(\frac{1}{2}\delta \hat{f}\left(X, \theta - 1\right) - \frac{\delta \hat{K}_{X'}\left|\hat{\Psi}\left(X'\right)\right|^2}{\hat{K}_{X'}\left|\hat{\Psi}\left(X'\right)\right|^2}\right)\hat{S}\left(X\right)\end{aligned}$$

$$\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\simeq-\frac{2S(X)}{\hat{f}(X,\theta-1)}$$

$$\delta\frac{\partial_{\hat{f}(X)}\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)S(X)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\simeq-2\frac{\frac{\hat{h}(X)}{2}\delta f(X,\theta-1)S(X)-2S(X)\frac{\delta\hat{f}(X,\theta-1)}{\hat{f}(X,\theta-1)}-\frac{\frac{\partial f(X)}{\partial S(X,\theta-2)}}{f(X)+C_0}\delta S(X,\theta-2)}{\hat{f}(X,\theta-1)}$$

$$\begin{aligned}\delta\alpha &\simeq \left(\frac{\hat{S}(X)}{\left(1-\left(\hat{S}(X)\right)\right)}\frac{1}{2\left(1+\frac{\Delta\hat{f}(X')+\Delta\hat{r}(X')}{2}\right)}\right.\\ &\left.-\frac{\hat{S}_1(X)}{\left(1-\left(\hat{S}_1(X')\right)\right)}\frac{1}{1+\Delta\hat{f}(X)}\right)\delta\hat{f}(X)\end{aligned}\tag{206}$$

In all expressions, due to presence of banks, $\hat{f}(X)$ stands for:

$$\hat{f}(X)+\frac{\left\langle\hat{S}(X',X)\right\rangle_{X'}\left\langle\hat{f}\right\rangle}{1-\left\langle\hat{S}\right\rangle}+\frac{\left(\left\langle\hat{S}_1^B(X,X)\right\rangle_X+\left\langle\hat{S}_2^B(X,X)\right\rangle_X+\left\langle\hat{S}(X',X)\right\rangle_{X'}\frac{\left\langle\hat{S}_1^B\right\rangle+\left\langle\hat{S}_2^B\right\rangle}{1-\left\langle\hat{S}\right\rangle}\right)\frac{\left\langle\bar{K}\right\rangle\left\|\bar{\Psi}\right\|^2}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\left\langle\bar{f}\right\rangle}{1-\left\langle\bar{S}\right\rangle}$$

We obtain:

$$\begin{aligned}\delta\alpha_0 &\simeq \left(\frac{\hat{S}_1(X)}{\left(1-\left(\hat{S}_1(X')\right)\right)}-\frac{\hat{S}(X)}{\left(1-\left(\hat{S}(X)\right)\right)}\right)\delta\hat{f}(X)\\ &+\delta\left\{\frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)}S_1(X,X,\theta-1)S(X)\right\}\\ &= \left(\frac{\hat{S}_1(X)}{\left(1-\left(\hat{S}_1(X')\right)\right)}-\frac{\hat{S}(X)}{\left(1-\left(\hat{S}(X)\right)\right)}\right)\delta\hat{f}(X)\\ &+\frac{\hat{S}(X)}{2\left(1-\hat{S}(X)\right)^2}\left(\frac{1}{2}\delta\hat{f}(X,\theta-1)-\frac{\delta\hat{K}_{X'}\left|\hat{\Psi}(X')\right|^2}{\hat{K}_{X'}\left|\hat{\Psi}(X')\right|^2}\right)S_1(X,X,\theta-1)S(X)\\ &+\frac{1-\left(\hat{S}_1(X)\right)}{1-\left(\hat{S}(X)\right)}\delta\left(S_1(X,X,\theta-1)S(X)\right)\end{aligned}\tag{207}$$

$$\begin{aligned}&\delta\left(S_1(X,X,\theta-1)S(X)\right)\\ \rightarrow\quad&\left(\frac{h(X)}{2}\left(\hat{h}(X)+\frac{h(X)}{2}\right)+\frac{\hat{h}(X)}{2}\right)S_1(X,X)S(X)\frac{\partial f(X)}{\partial S(X,\theta-2)}\delta S(X,\theta-2)\\ &+S(X)S_1(X)\frac{\delta\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\end{aligned}$$

$$\delta\alpha_0 \simeq \left(\frac{\hat{S}_1(X)}{\left(1-\left(\hat{S}_1(X')\right)\right)} - \frac{\hat{S}(X)}{\left(1-\left(\hat{S}(X)\right)\right)} + \frac{1}{2}\frac{\hat{S}(X)S_1(X,X)S(X)}{2\left(1-\hat{S}(X)\right)^2}\right)\delta\hat{f}(X) \tag{208}$$

$$-\frac{\hat{S}(X)S_1(X,X,\theta-1)S(X)}{2\left(1-\hat{S}(X)\right)^2}\frac{\delta\hat{K}_{X'}\left|\hat{\Psi}(X')\right|^2}{\hat{K}_{X'}\left|\hat{\Psi}(X')\right|^2}$$

$$+\frac{1-\left(\hat{S}_1(X)\right)S_1(X,X)S(X)}{1-\left(\hat{S}(X)\right)}\left(\frac{h(X)}{2}\left(\hat{h}(X)+\frac{h(X)}{2}\right)+\frac{\hat{h}(X)}{2}\right)\frac{\partial f(X)}{\partial S(X,\theta-2)}\delta S(X,\theta-2)$$

$$+\frac{1-\left(\hat{S}_1(X)\right)S_1(X,X)S(X)}{1-\left(\hat{S}(X)\right)}\frac{\delta\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}$$

$$\delta\alpha_1 \simeq -\frac{1}{2}\frac{\hat{S}(X)S_1(X,X)S(X)}{2\left(1-\hat{S}(X)\right)^2}\delta\hat{f}(X) \tag{209}$$

$$+\frac{\hat{S}(X)S_1(X,X,\theta-1)S(X)}{2\left(1-\hat{S}(X)\right)^2}\frac{\delta\hat{K}_{X'}\left|\hat{\Psi}(X')\right|^2}{\hat{K}_{X'}\left|\hat{\Psi}(X')\right|^2}$$

$$-\frac{1-\left(\hat{S}_1(X)\right)S_1(X,X)S(X)}{1-\left(\hat{S}(X)\right)}\left(\frac{h(X)}{2}\left(\hat{h}(X)+\frac{h(X)}{2}\right)+\frac{\hat{h}(X)}{2}\right)\frac{\partial f(X)}{\partial S(X,\theta-2)}\delta S(X,\theta-2)$$

$$-\frac{1-\left(\hat{S}_1(X)\right)S_1(X,X)S(X)}{1-\left(\hat{S}(X)\right)}\frac{\delta\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}\right)}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}$$

$$\begin{aligned}\delta\beta &\simeq \frac{\hat{h}(X)}{2}\delta S(X)\frac{\partial f(X)}{\partial S(X,\theta-2)}+\frac{\hat{h}(X)}{2}S(X)\frac{\partial^2 f(X)}{\partial S^2(X,\theta-2)}\delta S(X,\theta-2)\\ &\simeq \frac{\hat{h}(X)}{2}S(X)\left(\frac{\hat{h}(X)}{2}\left(\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)^2+\frac{\partial^2 f(X)}{\partial S^2(X,\theta-2)}\right)\delta S(X,\theta-2)\\ &\quad+\frac{\hat{h}(X)}{2}S(X)\frac{\partial f(X)}{\partial S(X,\theta-2)}\frac{\delta\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\end{aligned}$$

$$\delta\beta_0 \simeq \frac{\hat{h}(X)}{2}S(X)\frac{\partial f(X)}{\partial S(X,\theta-2)}\delta S(X,\theta-2)+\frac{\delta\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}S(X)$$

$$\delta h = \frac{\hat{h}(X)}{2}S(X)\frac{\partial f(X)}{\partial S(X,\theta-2)}\delta S(X,\theta-2)+\frac{\delta\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}S(X)$$

$$\delta \qquad c \simeq -\frac{\left(1-\left\langle \hat{S}\left(X'\right)\right\rangle\right)}{1-\left\langle \hat{S}_1\left(X'\right)\right\rangle}\frac{\left\langle \hat{h}\left(X',X\right)\right\rangle \left\langle h\left(X\right)\right\rangle}{4}\frac{\partial f\left(X\right)}{\partial S\left(X\right)}\delta \hat{f}\left(X'\right)$$

$$+\frac{\frac{h(X)}{4}\left(1+\hat{h}\left(X\right)\right)S_1\left(X\right)}{1+\left(\hat{h}\left(X\right)\left(f\left(X\right)-\frac{\left\langle \hat{f}(X')\right\rangle_{\hat{h}_1}+\left\langle \hat{r}(X')\right\rangle_{\hat{h}_2}}{2}\right)+\frac{h(X)}{2}\left(f\left(X\right)-\bar{r}\left(X\right)\right)\right)}\left(\frac{\partial f\left(X\right)}{\partial S\left(X\right)}\right)^2\delta S\left(X,\theta-2\right)$$

$$+\delta\left(\frac{\hat{h}\left(X\right)}{2}S_1\left(X\right)S\left(X\right)\frac{\partial f\left(X\right)}{\partial S\left(X,\theta-2\right)}\frac{\partial f\left(X\right)}{\partial S\left(X,\theta-1\right)}\right)$$

$$\delta \qquad c \simeq -\frac{\left(1-\left\langle \hat{S}\left(X'\right)\right\rangle\right)}{1-\left\langle \hat{S}_1\left(X'\right)\right\rangle}\frac{\left\langle \hat{h}\left(X',X\right)\right\rangle \left\langle h\left(X\right)\right\rangle}{4}\frac{\partial f\left(X\right)}{\partial S\left(X\right)}\delta \hat{f}\left(X'\right)$$

$$+\frac{\frac{h(X)}{4}\left(1+\hat{h}\left(X\right)\right)S_1\left(X\right)}{1+\left(\hat{h}\left(X\right)\left(f\left(X\right)-\frac{\left\langle \hat{f}(X')\right\rangle_{\hat{h}_1}+\left\langle \hat{r}(X')\right\rangle_{\hat{h}_2}}{2}\right)+\frac{h(X)}{2}\left(f\left(X\right)-\bar{r}\left(X\right)\right)\right)}\left(\frac{\partial f\left(X\right)}{\partial S\left(X\right)}\right)^2\delta S\left(X,\theta-2\right)$$

$$+\frac{\hat{h}\left(X\right)}{2}S_1\left(X\right)S\left(X\right)\left(\frac{\partial f\left(X\right)}{\partial S\left(X\right)}\right)^3$$

$$\times\left(\left(\frac{h\left(X\right)}{2}\left(\hat{h}\left(X\right)+\frac{h\left(X\right)}{2}\right)+\frac{\hat{h}\left(X\right)}{2}+\frac{\frac{\partial^2 f(X)}{\partial S^2(X)}}{\left(\frac{\partial f(X)}{\partial S(X)}\right)^2}\right)+\frac{\delta\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\right)}{\frac{\partial f(X)}{\partial S(X)}\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}}\right)\delta S\left(X,\theta-2\right)$$

$$\begin{aligned}\delta c_0 \quad &= \quad S_1\left(X\right)S\left(X\right)\left(\frac{\partial f\left(X\right)}{\partial S\left(X\right)}\right)^2 \\ &\times\left(\left(\frac{h\left(X\right)}{2}\left(\hat{h}\left(X\right)+\frac{h\left(X\right)}{2}\right)+\frac{\hat{h}\left(X\right)}{2}+\frac{\frac{\partial^2 f(X)}{\partial S^2(X)}}{\left(\frac{\partial f(X)}{\partial S(X)}\right)^2}\right)+\frac{\delta\left(\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}\right)}{\frac{\partial f(X)}{\partial S(X)}\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X\left|\Psi(X)\right|^2}}\right)\delta S\left(X,\theta-2\right)\end{aligned}$$

$$\frac{\partial_{\hat{f}(X,\theta-1)}\hat{K}_X\left|\hat{\Psi}\left(X\right)\right|^2}{\hat{K}_X\left|\hat{\Psi}\left(X\right)\right|^2}\simeq-\frac{2}{\hat{f}\left(X\right)}$$

$$\frac{\partial_{f(X)}K_X\left|\Psi\left(X\right)\right|^2}{K_X\left|\Psi\left(X\right)\right|^2}\simeq\frac{2}{r}\frac{1}{f\left(X\right)+C_0}$$

$$\delta\frac{\partial_{\hat{f}(X,\theta-1)}\hat{K}_X\left|\hat{\Psi}\left(X\right)\right|^2}{\hat{K}_X\left|\hat{\Psi}\left(X\right)\right|^2}=\frac{2}{\left(\hat{f}\left(X\right)\right)^2}\delta\hat{f}\left(X\right)$$

$$\begin{aligned}&\delta\left(\frac{\partial_{f(X)}K_X\left|\Psi\left(X\right)\right|^2}{K_X\left|\Psi\left(X\right)\right|^2}\frac{\partial f\left(X\right)}{\partial S\left(X,\theta-2\right)}\right) \\ \simeq \quad & -\left(1-\frac{\left(f\left(X\right)+C_0\right)\frac{\partial^2 f(X)}{\partial S^2(X)}}{\left(\frac{\partial f(X)}{\partial S(X)}\right)^2}\right)\frac{2}{r}\frac{\left(\frac{\partial f(X)}{\partial S(X)}\right)^2}{\left(f\left(X\right)+C_0\right)^2}\delta S\left(X,\theta-2\right)\end{aligned}$$

As before, the derivatives of these coefficients with respect to $\hat{K}_X \left|\hat{\Psi}(X)\right|^2$ and $K_X |\Psi(X)|^2$ can be neglected.

The matrices can then be obtained:

$$\hat{M} \simeq \begin{bmatrix} \frac{\partial}{\partial \hat{f}(X)}\alpha & \frac{\hat{h}(X',X)}{2}\frac{\partial}{\partial \hat{f}(X)}c & \frac{\partial}{\partial \hat{f}(X)}(\alpha_0 + c_0) & \frac{\partial}{\partial \hat{f}(X)}(\alpha_1 - c_0) \\ \frac{\partial}{\partial S(X)}\alpha & \frac{\hat{h}(X',X)}{2}\frac{\partial}{\partial S(X)}c & \frac{\partial}{\partial S(X)}(\alpha_0 + c_0) & \frac{\partial}{\partial S(X)}(\alpha_1 - c_0) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_S = \begin{bmatrix} 0 & \frac{\partial}{\partial \hat{f}(X)}\beta & \frac{\partial}{\partial \hat{f}(X)}(\beta_0 + h) & -\frac{\partial}{\partial \hat{f}(X)}(\beta_0 + h) \\ 0 & \frac{\partial}{\partial S(X)}\beta & \frac{\partial}{\partial S(X)}(\beta_0 + h) & -\frac{\partial}{\partial S(X)}(\beta_0 + h) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{\hat{K}} = \begin{bmatrix} \frac{\partial_{\hat{f}(X,\theta-1)}\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{\hat{K}_X\left|\hat{\Psi}(X)\right|^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \begin{array}{c} \partial_{f(X)}\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\left(\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)^2 \\ +\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\frac{\partial^2 f(X)}{\partial(S(X,\theta-2))^2} \end{array} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A5.2.4 Conditions for stability The condition for stability becomes:

$$\begin{aligned}
&\left(1-\langle\lambda^2\rangle\right)\left(\left(\delta\hat{S}(X)\right)^2 + (\delta S(X))^2 + \left(\delta\hat{K}_X\left|\hat{\Psi}(X)\right|^2\right)^2 + \left(\delta K_X|\Psi(X)|^2\right)^2\right) \\
> &\left\{\left(\delta\hat{S}(X), \delta S(X)\right)\left(\delta\hat{S}(X)\begin{pmatrix}\frac{\partial}{\partial \hat{f}(X)} \\ \frac{\partial}{\partial S(X)}\end{pmatrix}\alpha + \frac{\hat{h}(X',X)}{2}\delta S(X)\begin{pmatrix}\frac{\partial}{\partial \hat{f}(X)} \\ \frac{\partial}{\partial S(X)}\end{pmatrix}c\right.\right. \\
&\left.\left. + \frac{\delta\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\begin{pmatrix}\frac{\partial}{\partial \hat{f}(X)} \\ \frac{\partial}{\partial S(X)}\end{pmatrix}(\alpha_0 + c_0) + \frac{\delta K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\begin{pmatrix}\frac{\partial}{\partial \hat{f}(X)} \\ \frac{\partial}{\partial S(X)}\end{pmatrix}(\alpha_1 + \alpha_0)\right)\right\}^2 \\
&+\left\{\left(\delta\hat{S}(X), \delta S(X)\right)\left(\delta S(X)\begin{pmatrix}\frac{\partial}{\partial \hat{f}(X)} \\ \frac{\partial}{\partial S(X)}\end{pmatrix}\beta + \frac{\delta\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}{\frac{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{K_X|\Psi(X)|^2}}\begin{pmatrix}\frac{\partial}{\partial \hat{f}(X)} \\ \frac{\partial}{\partial S(X)}\end{pmatrix}(\beta_0 + h)\right)\right\}^2 \\
&+\left\{\partial_{\hat{f}(X)}\left(\frac{\partial_{\hat{f}(X)}\hat{K}_X\left|\hat{\Psi}(X)\right|^2}{\hat{K}_X\left|\hat{\Psi}(X)\right|^2}\right)\left(\delta\hat{S}(X)\right)^2\right\}^2 \\
&+\left\{\left(\partial_{f(X)}\left(\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\right)\left(\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\right)^2 + \frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}\right)(\delta S(X))^2\right\}^2
\end{aligned}$$

The dominant contributions involving the fluctuations in $\hat{K}_X \left|\hat{\Psi}(X)\right|^2$ and $K_X |\Psi(X)|^2$ are:

$$\left\{\left(\delta\hat{S}(X),\delta S(X)\right)\left(\delta S(X)\left(\begin{array}{c}\frac{\partial}{\partial\hat{f}(X)}\\\frac{\partial}{\partial S(X)}\end{array}\right)\beta+\frac{\delta\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}{\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}\left(\begin{array}{c}\frac{\partial}{\partial\hat{f}(X)}\\\frac{\partial}{\partial S(X)}\end{array}\right)(\beta_0+h)\right)\right\}^2$$

$$\simeq\left(\frac{\hat{h}(X)}{2}S(X)\right)^2\left\{\left(\frac{\hat{h}(X)}{2}\left(\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)^2+\frac{\partial^2 f(X)}{\partial S^2(X,\theta-2)}\right)\delta S(X)\right.$$

$$\left.+\frac{\partial f(X)}{\partial S(X,\theta-2)}\frac{\delta\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}{\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}\right\}^2$$

$$\left\{\left(\partial_{f(X)}\left(\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\right)\left(\frac{\partial f(X)}{\partial S^T(X,\theta-2)}\right)^2+\frac{\partial_{f(X)}K_X|\Psi(X)|^2}{K_X|\Psi(X)|^2}\frac{\partial^2 f(X)}{\partial\left(S^T(X,\theta-2)\right)^2}\right)(\delta S(X))^2\right\}^2$$

$$\simeq\left\{\left(1-\frac{(f(X)+C_0)\frac{\partial^2 f(X)}{\partial S^2(X)}}{\left(\frac{\partial f(X)}{\partial S(X)}\right)^2}\right)\frac{2}{r}\frac{\left(\frac{\partial f(X)}{\partial S(X)}\right)^2}{(f(X)+C_0)^2}\right\}^2\left((\delta S(X))^2\right)^2$$

This leads to the following condition:

$$\left(1-\langle\lambda^2\rangle\right)\left(\left(\delta\hat{S}(X)\right)^2+(\delta S(X))^2+\left(\delta\hat{K}_X\left|\hat{\Psi}(X)\right|^2\right)^2+\left(\delta K_X|\Psi(X)|^2\right)^2\right)$$

$$>A\left((\delta S(X))^2\right)^2+\left(B_1\delta S(X)+B_2\frac{\delta\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}{\frac{\hat{K}_X|\hat{\Psi}(X)|^2}{K_X|\Psi(X)|^2}}\right)^2$$

$$A=\left(1-\frac{(f(X)+C_0)\frac{\partial^2 f(X)}{\partial S^2(X)}}{\left(\frac{\partial f(X)}{\partial S(X)}\right)^2}\right)\frac{2}{r}\frac{\left(\frac{\partial f(X)}{\partial S(X)}\right)^2}{(f(X)+C_0)^2}$$

$$B_1=\frac{\hat{h}(X)}{2}S(X)\left(\frac{\hat{h}(X)}{2}\left(\frac{\partial f(X)}{\partial S(X,\theta-2)}\right)^2+\frac{\partial^2 f(X)}{\partial S^2(X,\theta-2)}\right)$$

$$B_2=\frac{\hat{h}(X)}{2}S(X)\frac{\partial f(X)}{\partial S(X,\theta-2)}$$